By Castigliano's theorem

$$\frac{dU}{dx} = -\frac{(P-X)l^3}{48EI} + \frac{Xa^3}{48EI} = 0$$

or $$X = \frac{Pl^3}{l^3 + a^3}$$

Example 4.4 Three bars each of length l and pinned at their ends are arranged in a vertical plane. (Fig. 4.8).

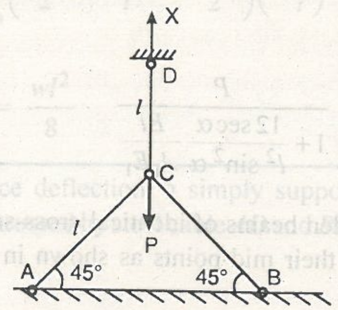


Fig. 4.8

The vertical bar has a cross sectional area A and each inclined bar has a cross sectional area A_1. *The* vertical load P acts at joint C and it is desired to find the ratio A_1/A to make the tension in DC numerically equal to the compressive forces in AC and BC.

Solution. Let X represent the tensile force in DC, chosen as the redundant bar. The compressive force in each inclined bar will be $(P-X)/\sqrt{2}$. Thus the strain energy of the system

$$U = \frac{X^2 l}{2AE} + \frac{(P-X)^2 l}{2A_1 E}$$

In this case, the end D of the vertical bar must have a displacement equal to zero, hence from the Castigliano's theorem,

$$\frac{dU}{dX} = \frac{Xl}{AE} - \frac{(P-X)l}{A_1 E} = 0$$

$$\Rightarrow \quad X = \frac{P}{1 + \frac{A_1}{A}} \qquad \text{...(1)}$$

The statement of the problem requires that

$$X = \frac{P-X}{\sqrt{2}} \qquad \text{...(2)}$$

Eliminating X between Eqn. (1) and (2),

$$\frac{A_1}{A} = \sqrt{2}$$

Example 4.5 A continuous beam of two equal spans L is uniformly loaded over its entire length. Find the magnitude R of the middle reaction by using the Castigliano's theorem.

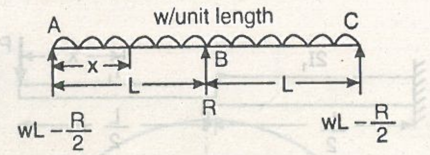


Fig. 4.9

Solution. Let R be the redundant reaction at B.

$$\therefore \quad \frac{\partial U_{AC}}{\partial R} = \frac{1}{EI}\int_A^C M\frac{\partial M}{\partial R}dx = 0 \qquad \text{...(1)}$$

The reactions at A and $C = \left(wL - \frac{R}{2}\right)$ each.

At any point distant x from A

$$M = -\left(wL - \frac{R}{2}\right)x + \frac{wx^2}{2}$$

$$\therefore \quad \frac{\partial M}{\partial R} = +\frac{x}{2}$$

Substituting the values in Eqn. (1),

$$\frac{2}{EI}\int_0^L \left\{-\left(wL - \frac{R}{2}\right)x + \frac{wx^2}{2}\right\}\frac{x}{2} = 0$$

or $$\left[-\left(wL - \frac{R}{2}\right)\frac{x^3}{6} + \frac{w}{4}\frac{x^4}{4}\right]_0^L = 0$$

$$\Rightarrow \quad -\frac{wL^4}{6} + \frac{RL^3}{12} + \frac{wL^4}{16} = 0$$

or $$R = \frac{5}{4}wL \quad \textbf{Ans.}$$

REVIEW QUESTIONS

Write short notes on the following :

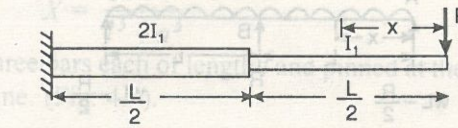
(*i*) Energy Method

(*ii*) Use of Energy Method to solve indeterminate beam problems

(*iii*) Castigliano's theorem

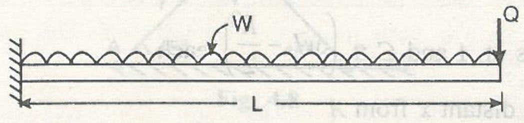
NUMERICAL PROBLEMS

1. A cantilever beam of stepwise constant cross section (see figure below), is loaded by a concentrated force at its tip. Determine the deflection under the point of application of the force by using Castigliano's theorem.



$$\left[\textbf{Hint. } \Delta = \int_0^{L/2} \frac{(Px)\,x\,dx}{E\,I_1} + \int_{L/2}^{L} \frac{(Px)\,x\,dx}{E(2\,I_1)}\right] \qquad \left[\textbf{Ans. } \frac{9PL^3}{48EI_1}\right]$$

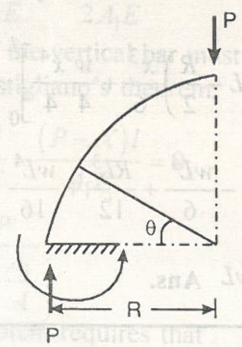
2. Use Castigliano's theorem to determine the deflection at the tip of a cantilever beam subjected to a uniformly distributed load of w per unit length.



$$\left[\textbf{Hint. } \Delta = \int_0^{L} \frac{\left[Qx + (wx^2/2)\right] x\,dx}{E\,I}\right] \qquad \left[\textbf{Ans. } \frac{w\,L^4}{8\,E\,I}\right]$$

where Q is auxiliary force, Put $Q = 0$]

3. A structure is in the form of one quadrant of a thin circular ring of radius R. One end is clamped and the other end is loaded by a vertical force P. Determine the vertical displacement under the point of application of the force P. Consider only strain energy of bending.

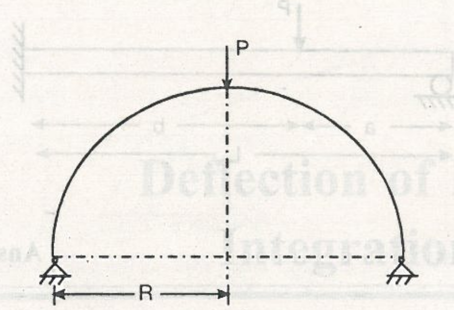


$$\left[\textbf{Ans. } \frac{P\,\pi\,R^3}{4\,E\,I}\right]$$

$$\left[\textbf{Hint. } \Delta = \int_0^{\pi/2} \frac{M(\partial M/\partial P)R d\theta}{EI} \quad \int_0^{\pi/2} \frac{(PR\cos\theta)(R\cos\theta)R d\theta}{EI}\right]$$

4. A thin semicircular ring is hinged at each end and located by a central concentrated force P. Determine the horizontal reaction at each hinge.

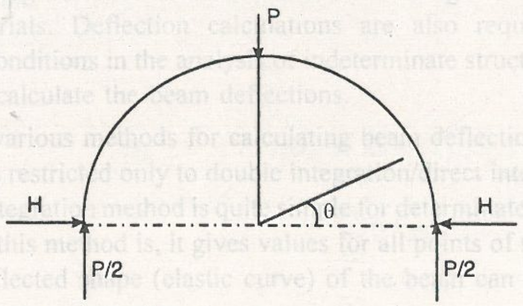
$$\left[\textbf{Ans. } \frac{P}{\pi}\right]$$



[**Hint** : Bending moment in the right half of the ring

$$M = \frac{P}{2}(R - R\cos\theta) - H R \sin\theta$$

$$\Delta_H = 0 = \frac{\Delta U}{\Delta H} = 2\int_0^{\pi/2} \frac{M\left(\dfrac{\delta M}{\delta H}\right) R\,d\theta}{E\,I}$$

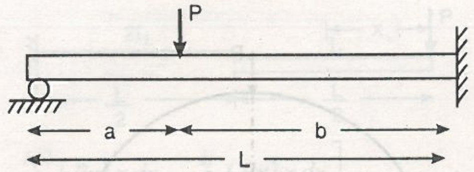


5. In Problem 4, determine the vertical displacement under the point of application of the central force P.

$$\left[\textbf{Ans. } \frac{P\,R^3}{E\,I}\left(\frac{3\pi}{8} + \frac{3}{2\,\pi} - 1\right)\right]$$

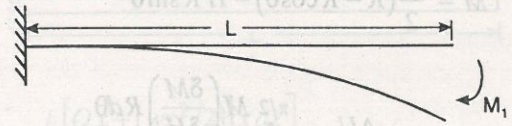
$$\left[\textbf{Hint. } \Delta = \frac{\partial U}{\partial P} = 2\int_0^{\pi/2} \frac{M\left(\dfrac{\partial M}{\partial H}\right) R\,d\theta}{EI}\right]$$

6. The beam shown below is supported at the left end, clamped at the right end and subjected to a concentrated load. Determine, the reaction at the left support by Castigliano's theorem.



$$\left[\textbf{Ans. } \frac{Pb^2(2L+a)}{2L^3} \right]$$

7. A cantilever beam is loaded by a moment M_1 applied at the tip. Determine by Castigliano's theorem the deflection of the tip.



$$\left[\textbf{Ans. } \frac{M_1 L^2}{2EI} \right]$$

5

Deflection of Beams by Integration Method

5.1 INTRODUCTION

Materials used for beams are elastic and hence under the action of loads the beam axes deflect. A designer has to decide about beam dimensions not only based on strength requirement but also from the consideration of deflections which should be within the prescribed limits.

In mechanical components excessive deflection may cause mis-alignment and non-performance of the machine. In buildings excessive deformation gives rise to psychological unrest and sometimes to breaking of flooring, ceiling or roofing materials. Deflection calculations are also required to impose consistency conditions in the analysis of indeterminate structures. Hence it is necessary to calculate the beam deflections.

There are various methods for calculating beam deflection. The scope of this chapter is restricted only to double integration/direct integration method. The double integration method is quite simple for determinate beams. Another advantage of this method is, it gives values for all points of the structure and hence the deflected shape (elastic curve) of the beam can be drawn.

5.2 DIFFERENTIAL EQUATION FOR DEFLECTION

Consider an elemental length $AB = ds$ as shown in Fig. 5.1. Let tangents drawn at A and B make angles θ and $\theta + d\theta$ with x-axis and intersect it at D and E.

Let M be intersection point of these two tangents.

$$\therefore \quad \angle DME = d\theta$$

Also we note that

$$\angle DME + \angle AMB = 180°$$

$$\text{But} \quad \angle AMB + \angle ACB = 360° - 90° - 90° = 180°$$

$$\therefore \quad \angle AMB + \angle ACB = \angle DME + AMB$$

$$ACB = DME = d\theta$$

$$ds = R\,d\theta \quad \ldots(1)$$

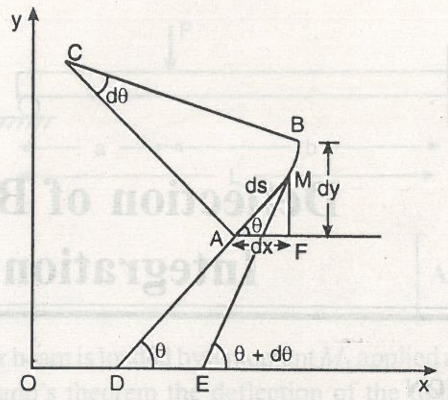


Fig. 5.1

Since ds is an elemental length, treating ABF as a triangle

$$\frac{ds}{dx} = \sec\theta \quad \ldots(2)$$

and

$$\frac{dy}{dx} = \tan\theta \quad \ldots(3)$$

From Eqn. (1)

$$\frac{1}{R} = \frac{d\theta}{ds} \quad \ldots(4)$$

Differentiating Eqn. (3) with respect to x, we get

$$\frac{d^2y}{dx^2} = \sec^2\theta\,\frac{d\theta}{dx}$$

$$= \sec^2\theta\,\frac{d\theta}{ds}\,\frac{ds}{dx}$$

$$= \sec^2\theta\,\frac{1}{R}\sec\theta$$

$$= \sec^3\theta \times \frac{1}{R}$$

$$\frac{1}{R} = \frac{d^2y/dx^2}{(1+\tan^2\theta)^{3/2}} \quad \text{since } \sec^2\theta = 1+\tan^2\theta$$

$$= \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

In beams, deflections are small and hence, slope dy/dx is small. Therefore, in this theory, which may be called small deflection theory, $(dy/dx)^2$ is neglected compared to unity and hence,

$$\frac{1}{R} = \frac{d^2y}{dx^2} \quad \ldots(6)$$

From the bending equation of beam, we know

$$\frac{M}{I} = \frac{E}{R}$$

or

$$\frac{1}{R} = \frac{M}{EI} \quad \ldots(7)$$

Hence

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

or

$$EI\frac{d^2y}{dx^2} = M \quad \ldots(8)$$

This equation is called differential equation for deflection. Note that the following sign conventions are used in deriving Eqn. (8)

(*i*) The y-axis is upward.

(*ii*) Curvature is concave towards the positive y-axis.

(*iii*) This type of curvature occurs in the beam due to the sagging moment. Hence, the sagging moment is to be considered as the positive moment.

In some text books, $EI\dfrac{d^2y}{dx^2} = -M$ is taken to get downward deflection positive when the sagging moment is taken as positive. In this book, the upward deflection and the sagging moments are taken as positive and hence, the equation used is

$$EI\frac{d^2y}{dx^2} = M$$

The term EI is called flexural rigidity.

5.3 OTHER USEFUL EQUATIONS

The differential relations relating to load, shear and moments [Eqns. (1) and (2)] can be clubbed with Eqn. (8) to get other useful differential equations

$$\text{Deflection} = y$$

$$\text{Slope } \theta = \frac{dy}{dx}$$

$$\text{Moment } M = EI\frac{d^2y}{dx^2}$$

$$\text{Shear force } F = -\frac{dM}{dx} = -EI\frac{d^3y}{dx^3}$$

$$\text{Load density } q = \frac{dF}{dx} = -EI\frac{d^4y}{dx^4}$$

5.4 INTEGRATION METHOD

In this method, the moment M, at any distance x from one of the supports (usually left hand support) is written with the sagging moment as positive.

Then from Eqn. (8), we have

$$EI\frac{d^2y}{dx^2} = M$$

$$\therefore \quad EI\frac{dy}{dx} = \int_0^x Mdx + C_1$$

and

$$EIY = \int_0^x \int_0^x Mdx + C_1x + C_2$$

The constants C_1 and C_2 are found by making use of boundary conditions. Useful conditions are listed below.

(*a*) At simply supported/roller ends

deflection $y = 0$

(*b*) At fixed ends

deflection $y = 0$ and slope $\frac{dy}{dx} = 0$

(*c*) At point of symmetry $\frac{dy}{dx} = 0$

5.5 A FEW GENERAL CASES

5.5.1 Cantilever Subjected to Moment at Free End

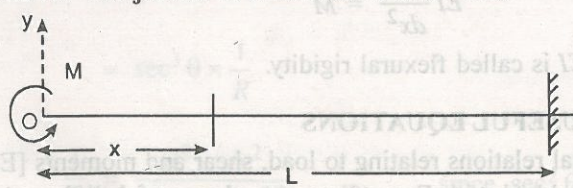


Fig. 5.2

Figure 5.2 shows a cantilever beam of span L, flexural rigidity EI, subjected to hogging moment M. Taking origin O at free end, moment at distance x is given by

$$M_x = -M \quad \text{...(1)}$$

i.e.

$$EI\frac{d^2y}{dx^2} = -M$$

$$\therefore \quad EI\frac{dy}{dx} = -Mx + C_1 \quad \text{...(2)}$$

and

$$EIy = -\frac{Mx^2}{2} + C_1x + C_2 \quad \text{...(3)}$$

The boundary conditions available are

$$\text{At } x = L \quad \frac{dy}{dx} = 0 \quad \text{...(4)}$$

$$\text{and} \quad y = 0 \quad \text{...(5)}$$

From boundary condition (4) and Eqn. (2), we get

$$0 = -ML + C_1$$

$$\text{or} \quad C_1 = ML \quad \text{...(6)}$$

From boundary condition (5) and Eqn. (3), we get

$$0 = -\frac{ML^2}{2} + C_1L + C_2$$

Substituting the value of C_1 and re-arranging

$$C_2 = \frac{ML^2}{2} - ML^2 = -\frac{ML^2}{2}$$

$\therefore$ From Eqn. (2) and (3), we get

$$EI\frac{dy}{dx} = -Mx + M_1L = M(L-x)$$

and

$$EI\,y = \frac{Mx^2}{2} - MLx - \frac{ML^2}{2}$$

$$= M\left[-\frac{x^2}{2} + Lx - \frac{L^2}{2}\right]$$

At free end, $x = 0$

$$\frac{dy}{dx} = \frac{1}{EI}ML = \frac{ML}{EI}$$

and

$$y = \frac{1}{EI}M\left(-\frac{L^2}{2}\right) = -\frac{ML^2}{2EI}$$

i.e.

$$y = \frac{ML^2}{2EI} \text{ downward}$$

5.5.2 Cantilever Subjected to Concentrated Load at Free End

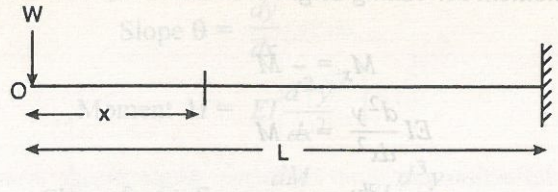


Fig. 5.3

Referring to Fig. 5.3 and taking hogging moment negative.

$$M_x = -Wx$$

i.e. $$EI\frac{d^2y}{dx^2} = -Wx$$

$\Rightarrow$ $$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + C_1$$

At $x = L$, $$\frac{dy}{dx} = 0$$

$$0 = -\frac{WL^2}{2} + C_1$$

$\Rightarrow$ $$C_1 = \frac{WL^2}{2}$$

$\therefore$ $$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + \frac{WL^2}{2}$$

Integrating $$EIy = -\frac{Wx^3}{6} + \frac{WL^2}{2}x + C_2$$

The boundary condition is,

At $x = L$, $$y = 0$$

$$0 = -\frac{WL^3}{6} + \frac{WL^2}{2}x + C_2$$

$\Rightarrow$ $$C_2 = -WL^3\left(\frac{1}{2} - \frac{1}{6}\right)$$

$$= -\frac{WL^3}{3}$$

$\therefore$ $$EIy = -\frac{Wx^3}{6} + \frac{WL^2}{2}x - \frac{WL^3}{3}$$

At free end $x = 0$

$$\frac{dy}{dx} = \frac{1}{EI}\left(\frac{WL^2}{2}\right) = \frac{WL^2}{2EI}$$

and $$y = \frac{1}{EI}\left(-\frac{WL^3}{3}\right) = -\frac{WL^3}{3EI}$$

i.e. $\dfrac{WL^3}{3EI}$ downward.

5.5.3 A Cantilever Subjected to Uniformly Distributed Load

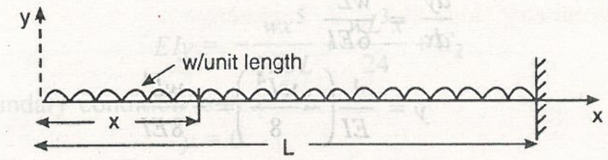


Fig. 5.4

Referring to Fig. 5.4 and taking hogging moment negative

$$M_x = -\frac{wx^2}{2}$$

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}$$

$\therefore$ $$EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1$$

At $x = L$, $$\frac{dy}{dx} = 0$$

$\therefore$ $$0 = -\frac{wL^3}{6} + C_1$$

$\Rightarrow$ $$C_1 = \frac{wL^3}{6}$$

$\therefore$ $$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wL^3}{6}$$

Integrating again, we get

$$EIy = -\frac{wx^4}{24} + \frac{wL^3x}{6} + C_2$$

At $x = L$, $y = 0$

$$\therefore \quad 0 = -\frac{wL^4}{24} + \frac{wL^4}{6} + C_2$$

or
$$C_2 = -\frac{wL^4}{6} + \frac{wL^4}{24}$$

$$= -\frac{wL^4}{8}$$

$$\therefore \quad EIy = -\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}$$

At free end where $x = 0$, we get

$$\frac{dy}{dx} = \frac{wL^3}{6EI}$$

and
$$y = \frac{1}{EI}\left(-\frac{wL^4}{8}\right) = -\frac{wL^4}{8EI}$$

$$\Rightarrow \quad y = \frac{wL^4}{8EI} \text{ downward.}$$

5.5.4 A Cantilever Subjected to have Varying Linearly from Zero at Free End to w/unit Length at Fixed End

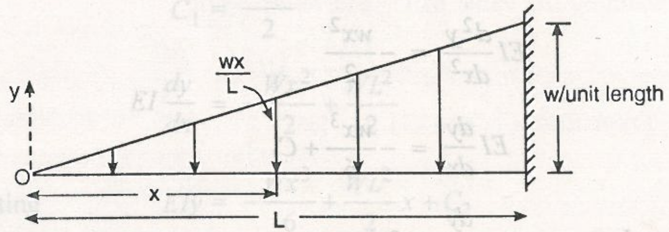


Fig. 5.5

Consider a section at distance x from free end as shown in Fig. 5.5. Here intensity of loading is wx/L and its C.G. is at $x/3$ from the section.

Hence
$$M_x = -\frac{x}{2}\frac{wx}{L}\frac{x}{3} = -\frac{wx^3}{6L}$$

i.e.
$$EI\frac{d^2y}{dx^2} = -\frac{wx^3}{6L}$$

Integrating with respect to x, we get

$$EI\frac{dy}{dx} = -\frac{wx^4}{24L} + C_1$$

The boundary condition is at $x = L$

$$\frac{dy}{dx} = 0$$

$$0 = -\frac{wL^4}{24L} + C_1$$

or
$$C_1 = \frac{wL^3}{24}$$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{wx^4}{24L} + \frac{wL^3}{24}$$

Integrating again, we get

$$EIy = -\frac{wx^5}{120L} + \frac{wL^3x}{24} + C_2$$

The boundary condition is at $x = L$

$$y = 0$$

i.e.
$$0 = -\frac{wL^5}{120L} + \frac{wL^3L}{24} + C_2$$

$$C_2 = \frac{wL^4}{120} - \frac{wL^4}{24} = \frac{wL^4}{120}(1-5) = -\frac{wL^4}{30}$$

$$\therefore \quad EIy = \frac{-wx^5}{120L} + \frac{wL^3}{24}x - \frac{wL^4}{30}$$

At free end where $x = 0$, we get

$$\frac{dy}{dx} = \frac{1}{EI}\cdot\frac{-wL^4}{24} = \frac{wL^4}{24EI}$$

$$y = \frac{1}{EI}\left(\frac{-wL^4}{30}\right) = \frac{-wL^4}{30EI}$$

i.e.
$$y = \frac{wL^4}{30EI} \text{ downward}$$

5.5.5 Simply Supported Beam Subjected to a Central Concentrated Load

Consider the simply supported beam AB of span L carrying central concentrated load W at C, the centre of its span. (Fig. 5.6)

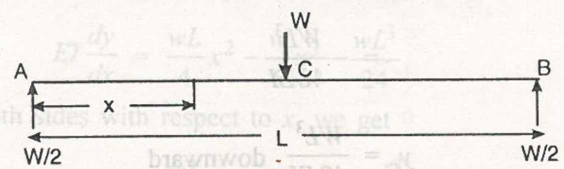


Fig. 5.6

$$\text{Reaction } R_A = \frac{W}{2}$$

$$M_x = R_A x = \frac{Wx}{2}$$

or $$EI\frac{d^2y}{dx^2} = \frac{Wx^2}{2}$$

$\therefore$ $$EI\frac{dy}{dx} = \frac{Wx^2}{4} + C_1$$

Due to symmetry slope at $x = L/2$ is zero

$$0 = \frac{W(L/2)^2}{4} + C_1$$

or $$C_1 = -\frac{WL^2}{16}$$

$\therefore$ $$EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL^2}{16}x + C_2$$

At $x = 0$, $y = 0$

$\therefore$ $0 = C_2$

Hence $$EIy = \frac{Wx^3}{12} - \frac{WL^2}{16}x$$

$\therefore$ Deflection at mid span *i.e.* at $x = \frac{L}{2}$ is

$$y_c = \frac{1}{EI}\left[\frac{W}{12}\left(\frac{L}{2}\right)^3 - \frac{WL^2}{16}\frac{L}{2}\right]$$

$$= \frac{WL^3}{EI}\left[\frac{1}{96} - \frac{1}{32}\right]$$

$$= -\frac{WL^3}{48EI}$$

$\therefore$ $$y_c = \frac{WL^3}{48EI} \text{ downward}$$

Slope at support is obtained by putting $x = 0$ in slope equation

$\therefore$ $$\theta_A = \left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{EI}\left(-\frac{WL^2}{16}\right)$$

$$= -\frac{WL^2}{16EI}$$

5.5.6 Simply Supported Beam Subjected to Uniformly Distributed Load

Let AB be the simply supported beam of span L, subjected to uniformly distributed load w/unit length through out as shown in Fig. 5.7.

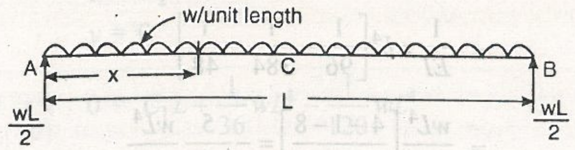


Fig. 5.7

$$R_A = R_B = \frac{wL}{2}$$

$\therefore$ $$M_x = \frac{wL}{2}x - \frac{wx^2}{2}$$

i.e. $$EI\frac{d^2y}{dx^2} = \frac{wL}{2}x - \frac{wx^2}{2}$$

$\therefore$ $$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1$$

Due to symmetry $\frac{dy}{dx} = 0$ at $x = \frac{L}{2}$

$\therefore$ $$0 = \frac{wL}{4}\left(\frac{L}{2}\right)^2 - \frac{w}{6}\left(\frac{L}{2}\right)^3 + C_1$$

or $$C_1 = wL^3\left[-\frac{1}{16} + \frac{1}{48}\right] = -\frac{wL^3}{24}$$

$\therefore$ $$EI\frac{dy}{dx} = \frac{wL}{4}x^2 - \frac{w}{6}x^3 - \frac{wL^3}{24}$$

Integrating both sides with respect to x, we get

$$EIy = \frac{wL}{12}x^3 - \frac{w}{24}x^4 - \frac{wL^3}{24}x + C_2$$

At $x = 0$, $\quad y = 0$

$$\therefore \quad 0 = C_2$$

Hence $$EIy = \frac{wL}{12}x^3 - \frac{w}{24}x^4 - \frac{wL^3}{24}x$$

$\therefore$ Maximum deflection y_c which occurs at centre C is obtained by substituting $x = L/2$ in the above equation.

$$y_c = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L}{2}\right)^3 - \frac{w}{24}\left(\frac{L}{2}\right)^4 - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$= \frac{1}{EI}wL^4\left[\frac{1}{96} - \frac{1}{384} - \frac{1}{48}\right]$$

$$= \frac{wL^4}{EI}\left[\frac{4-1-8}{384}\right] = -\frac{5}{384}\frac{wL^4}{EI}$$

or $$y_c = \frac{5}{384}\frac{wL^4}{EI} \text{ downward.}$$

Slope at end $$\theta_4 = \left(\frac{dy}{dx}\right)_{x=0} = \frac{wL^3}{24EI}$$

5.5.7 A Simply Supported Beam Subjected to a Load Varying Linearly from Zero at One End to *w*/unit Length at Other End

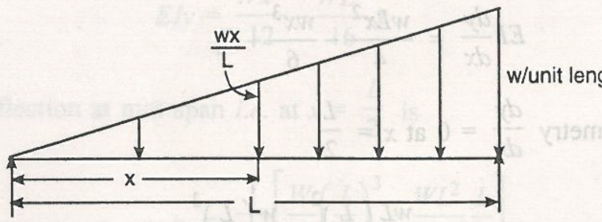


Fig. 5.8

Referring to Fig. 5.8

$$R_A L = \frac{1}{2}wL\frac{L}{3}$$

$$R_A = \frac{1}{6}wL$$

$$\therefore \quad M_x = R_A x - \frac{1}{2}x\frac{wx}{L}\frac{x}{3}$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = \frac{1}{6}wLx - \frac{1}{6}\frac{wx^3}{L}$$

$$\therefore \quad EI\frac{dy}{dx} = C_1 + \frac{1}{12}wLx^2 - \frac{1}{24}\frac{wx^4}{L}$$

and $$EIy = C_2 + C_1x + \frac{1}{36}wLx^3 - \frac{1}{120}\frac{wx^5}{L}$$

At $x = 0$, $\quad y = 0$

$$0 = C_2$$

At $x = L$, $\quad y = 0$

$$0 = C_1L + \frac{1}{36}wL^4 - \frac{1}{120}wL^4$$

$$C_1 = wL^3\left[-\frac{1}{36} + \frac{1}{120}\right]$$

$$= -\frac{7wL^3}{360}$$

$$\therefore \quad EIy = -\frac{7wL^3}{360} + \frac{wlx^3}{36} - \frac{wx^5}{120L}$$

and $$EI\frac{dy}{dx} = \frac{7}{360}wL^3 + \frac{wLx^2}{12} - \frac{wx^4}{24L}$$

At the point of maximum deflection $\frac{dy}{dx} = 0$

$$\therefore \quad 0 = -\frac{7}{360}wL^3 + \frac{wLx^2}{12} - \frac{wx^4}{24L}$$

or $$x^4 - 2L^2x^2 + \frac{7}{15}L^4 = 0$$

$$x^2 = \frac{2L^2 \pm \sqrt{4L^2 - \frac{4\times 7}{15}L^4}}{2}$$

$$= L^2\left(1 \pm \sqrt{1 - 7/15}\right)$$

$$= 0.2697\ L^2$$

or $$x = 0.5193\ L$$

$$\therefore \quad y_{max} = \frac{wL^4}{EI}\left[-\frac{7}{360} \times 0.5193 + \frac{0.5193^3}{36} - \frac{0.5193^5}{120}\right]$$

$$= 0.006523\frac{wL^4}{EI}$$

Thus the maximum deflection occurs at a distance 0.5193 L from the end with load intensity zero and its value is 0.006523 $\frac{wL^4}{EI}$ downward.

Example 5.1 A cantilever beam of span L is subjected to a concentrated load W at a distance 'a' from fixed end. Find the deflection of free end.

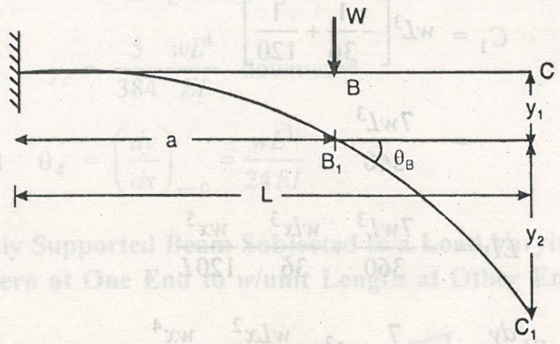


Fig. 5.9

Solution. Let AC be the cantilever subjected to load W at B as shown in Fig. 5.9. Let AB_1C_1 be the deflected shape.

Now, deflection at B, $$y_1 = \frac{Wa^3}{3EI}$$

and slope at B $$\theta_B = \frac{Wa^2}{2EI}$$

Since the portion BC is not subjected to any moment, it remains straight and its slope is θ_B.

$\therefore$ Deflection at C = Deflection at B + $(L - a)$ slope at B

$$= y_1 + (L-a)\theta_B$$

$$= \frac{Wa^3}{3EI} + (L-a)\frac{Wa^2}{2EI}$$

Example 5.2 Find the displacement at free end of the cantilever shown in Fig. 5.10. Find its numerical value if L = 3 m, a = 2 m, W_1 = 20 kN, W_2 = 30 kN, E = 2 × 10^5 N/mm², I = 2 × 10^8 mm⁴.

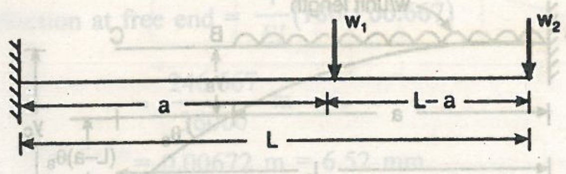


Fig. 5.10

Solution. Deflection at free end due to $W_2 = \frac{W_2L^3}{3EI}$

$$\text{Deflection at free end due to } W_1 = \frac{W_1a^3}{3EI} = +(L-a)\frac{W_1a^2}{2EI}$$

$$\therefore \quad \text{Total deflection at free end} = \frac{W_1a^3}{3EI} + (L-a)\frac{W_1a^2}{2EI} + \frac{W_2L^3}{3EI}$$

$\therefore$ Deflection at free end in given problem

$$= \frac{1}{EI}\left[\frac{20 \times 2^3}{3} + \frac{(3-2) \times 20 \times 2^2}{2} + \frac{30 \times 3^3}{3}\right]$$

$$= \frac{363.333}{EI}\ \text{m}$$

To get the numerical value correctly consistency of units should be used. If 'W' is in kN, 'a' and 'L' are in metres, 'E' and 'I' are also to be used in kN and m units.

$$E = 2 \times 10^5\ \text{N/mm}^2$$
$$I = 2 \times 10^8\ \text{mm}^4$$
$$EI = 2 \times 10^5 \times 2 \times 10^8\ \text{N-mm}^2$$
$$= 4 \times 10^{13}\ \text{N-mm}^2$$
$$= 4 \times 10^{13} \times 10^{-9}\ \text{kN-m}^2 = 4 \times 10^4\ \text{kN-m}^2$$

Note : when E and I both are in N and mm units, EI is in N-mm² unit. To convert it to kN-m² unit multiplying factor is 10^{-9}.

$$\therefore \quad \text{Deflection at free end} = \frac{363.333}{4 \times 10^4}\ \text{m}$$

$$= 0.009083\ \text{m}$$

$$= 9.083\ \text{mm}$$

Example 5.3 Find the deflection at free end in the cantilever beam, shown in Fig. 5.11.

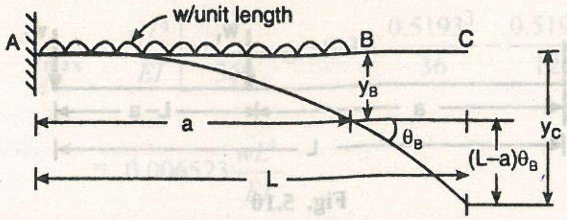


Fig. 5.11

Solution. Let ABC be the beam as shown in Fig. 5.11

Now $\quad y_c = y_B + (L-a)\theta_B$

Since BC remains straight with slope θ_A

$$y_c = \frac{Wa^4}{8EI} + (L-a)\frac{Wa^3}{6EI}$$

Example 5.4 Find the displacement of free end of cantilever beam shown in Fig. 5.12. Take $E = 2 \times 10^5$ N/mm², $I = 180 \times 10^6$ mm⁴

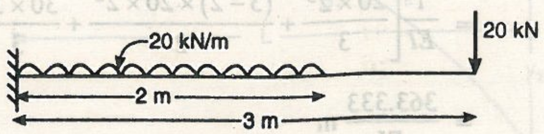


Fig. 5.12

Solution. Displacement of free end due to 20 kN concentrated load at free end.

$$= \frac{WL^3}{3EI} = \frac{20 \times 3^3}{3EI} = \frac{180}{EI}$$

Displacement of free end due to u.d.l.

$$= \frac{Wa^4}{8EI} + (L-a)\frac{Wa^3}{6EI}$$

$$= \frac{1}{EI}\left[\frac{20 \times 2^4}{8} + \frac{(3-2) \times 20 \times 2^3}{6}\right]$$

$$= \frac{1}{EI} \times 66.6667$$

Since loads are taken in kN, and a, L are taken in m units, EI is to be taken in kN-m² unit to get displacement in m units

$$EI = 2 \times 10^5 \times 180 \times 10^6 \times 10^{-9}$$

$$= 36000 \text{ kN-m}^2$$

$\therefore$ Deflection at free end $= \dfrac{1}{EI}(180 + 66.667)$

$$= \frac{246.667}{36000} \text{ m}$$

$$= 0.00672 \text{ m} = 6.52 \text{ mm}$$

Example 5.5 A cantilever of span L carries a uniformly distributed load w/unit length over a distance 'a' from free end. Find the expression for displacement of free end.

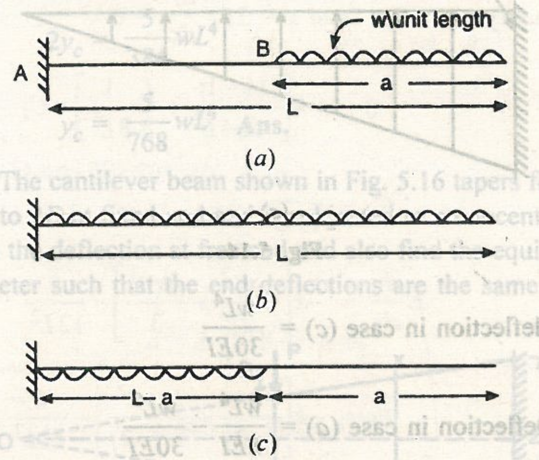


Fig. 5.13

Solution. Such a beam is shown in Fig. 5.13. This system is same as combination of loadings shown in (*b*) and (*c*)

$\therefore$ deflection at free end of beam in Fig. 5.13(*a*)

= Downward deflection at free end in beam shown in Fig. 5.13(*b*) minus upward deflection at free end in beam shown in Fig. 5.13(*c*).

$$= \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI}a\right]$$

Example 5.6 A cantilever beam is subjected to linearly varying load as shown in Fig. 5.14(*a*). Find the expression for deflection at free end.

Solution. This problem may be considered as combination of case (*b*) and case (*c*) as shown in Fig. 5.14(*b*) and 5.14(*c*).

Downward deflection of free end in case $B = \dfrac{wL^4}{8EI}$

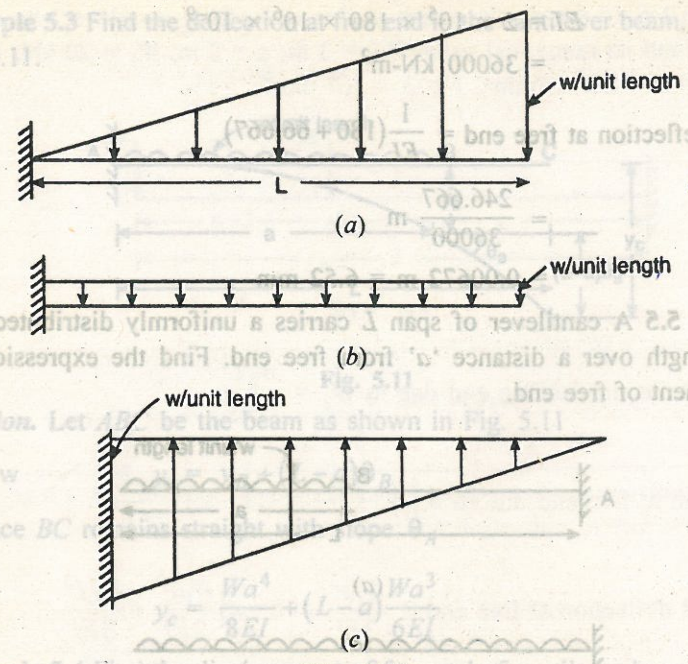


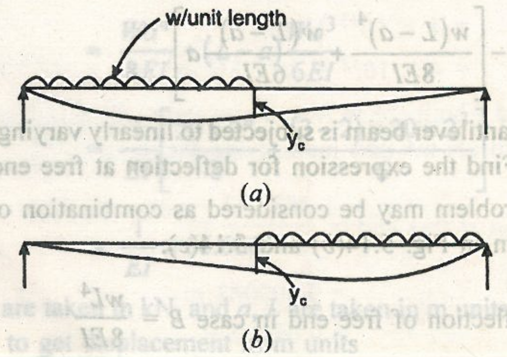
Fig. 5.14

$$\text{Upward deflection in case } (c) = \frac{wL^4}{30EI}$$

$$\therefore \quad \text{Deflection in case } (a) = \frac{wL^4}{8EI} - \frac{wL^4}{30EI}$$

$$= \frac{wL^4}{EI}\left[\frac{15-4}{120}\right] = \frac{11}{120}\frac{wL^4}{EI}$$

Example 5.7 A simply supported beam is subjected to uniformly distributed load in one half portion as shown in Fig. 5.15(*a*). Find the displacement at the centre of the span.



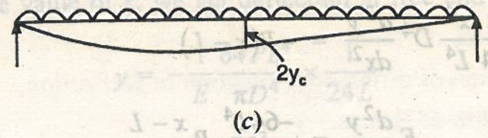


Fig. 5.15

Solution. Let the displacement for the given case be y_c. if other half is loaded instead of first half, Fig. 5.15(*b*) then also displacement = y_c.

$\therefore$ when both halves are loaded as shown in Fig. 5.15(*c*) displacement should be $2y_c$. But for this case

$$\text{Displacement} = \frac{5}{384}wL^4$$

$$2y_c = \frac{5}{384}wL^4$$

$$\Rightarrow \quad y_c = \frac{5}{768}wL^4 \quad \textbf{Ans.}$$

Example 5.8 The cantilever beam shown in Fig. 5.16 tapers from a diameter *D* at free end to 2*D* at fixed end and is subjected to a concentrated load *P* at free end. Find the deflection at free end and also find the equivalent beam of uniform diameter such that the end deflections are the same.

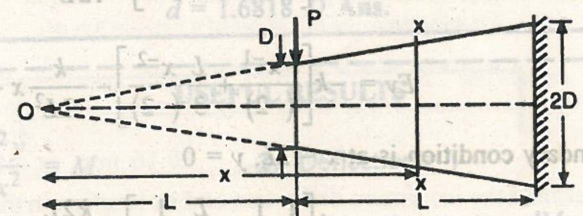


Fig. 5.16

Solution. Let *O* be the origin as shown in Fig. 5.16. At any distance *x*, the diameter of the beam.

$$\frac{x}{2L} \times 2D = \frac{x}{L} \times D$$

$$\therefore \quad I = \frac{\pi}{64}\left(\frac{x}{L}D\right)^4 = \frac{\pi}{64}\frac{x^4}{L^4}D^4$$

Moment at section *x-x* is

$$M = -P(x-L) \qquad \text{since it is hogging}$$

$$\text{i.e.} \quad EI\frac{d^2y}{dx^2} = -P(x-L)$$

i.e. $$E\frac{\pi}{64}\frac{x^4}{L^4}D^4\frac{d^2y}{dx^2} = -P(x-L)$$

⇒ $$E\frac{d^2y}{dx^2} = \frac{-64L^4}{\pi D^4}P\frac{x-L}{x^4}$$

$$= -k\left(x^{-3} - Lx^{-4}\right) \quad \text{where } K = \frac{64L^4}{\pi D^4}$$

∴ $$E\frac{dy}{dx} = -k\left[-\frac{1}{2}x^{-2} - \frac{Lx^{-3}}{-3}\right] + C_1$$

Now, boundary condition is at $x = 2L$, $\frac{dy}{dx} = 0$

$$0 = -k\left[-\frac{1}{2}\frac{1}{4L^2} + \frac{L}{3}\frac{1}{8L^3}\right] + C_1$$

∴ $$C_1 = \frac{-k}{12L^2}$$

∴ $$E\frac{dy}{dx} = -k\left[\frac{-x^{-2}}{2} + \frac{Lx^{-3}}{3}\right] - \frac{k}{12L^2}$$

∴ $$Ey = -k\left[\frac{-x^{-1}}{(-2)} + \frac{L}{3}\frac{x^{-2}}{(-2)}\right] - \frac{k}{12L^2}x + C_2$$

The boundary condition is at $x = 2L$, $y = 0$

$$0 = -k\left[\frac{1}{2}\frac{1}{2L} - \frac{L}{6}\frac{1}{4L^2}\right] - \frac{k2L}{12L^2} + C_2$$

or $$C_2 = \frac{k}{L}\left[\frac{1}{4} - \frac{1}{24} + \frac{1}{6}\right] = \frac{3}{8}\frac{k}{L}$$

∴ $$Ey = -k\left[\frac{x^{-1}}{2} - \frac{L}{6}x^{-1}\right] - \frac{kx}{12L^2} + \frac{3}{8}\frac{k}{L}$$

∴ At free end where $x = L$

$$Ey = -k\left[\frac{1}{2L} - \frac{1}{6L} + \frac{1}{12L} - \frac{3}{8L}\right]$$

$$= -\frac{k}{24L}(12 - 4 + 2 - 9) = -\frac{k}{24L}$$

Substituting the value of k, we get deflection at free end

$$y = -\frac{1}{E}\frac{64PL^4}{\pi D^4} \times \frac{1}{24L}$$

$$= -\frac{8}{3}\frac{PL^3}{\pi D^4 E}$$

$$= \frac{8}{3}\frac{PL^3}{\pi D^4 E} \text{ Downward} \quad \ldots(1)$$

If the diameter of equivalent beam of uniform section is d, then its downward deflection.

$$y = \frac{1}{EI}\frac{PL^3}{3}$$

$$= \frac{1}{\left(\frac{E\pi d^4}{64}\right)} \times \frac{PL^3}{3} = \frac{64}{3}\frac{PL^3}{\pi d^4 E} \quad \ldots(2)$$

From Eqn. (1) and (2), we get

$$\frac{8}{3D^4} = \frac{64}{3d^4}$$

⇒ $$d = \sqrt[4]{8}D$$

⇒ $$d = 1.6818\ D \text{ Ans.}$$

USEFUL RESULTS

1. $EI\frac{d^2y}{dx^2} = M$
2. Deflection $= y$
3. Slope $\theta = \frac{dy}{dx}$
4. Shear force $F = \frac{dM}{dx} = -EI\frac{d^3y}{dx^3}$
5. Load density $\frac{dF}{dx} = -EI\frac{d^4y}{dx^4}$
6. Integration Method

$$EI\frac{d^2y}{dx^2} = M$$

∴ $$EI\frac{dy}{dx} = \int_0^x M dx + C_1$$

and $$EIy = \int_0^x\int_0^x M dx + C_1 x + C_2$$

REVIEW QUESTIONS

Write short notes on following :

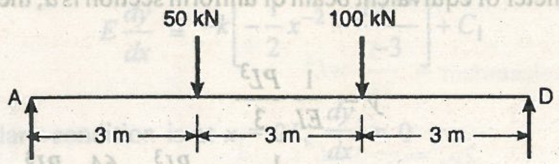
(*i*) Derivation of differential equation of Deflection.

(*ii*) Deflection of Beams.

(*iii*) Double Integration Method.

NUMERICAL PROBLEMS

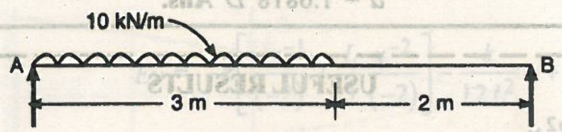
1. Determine the deflection at the point *B* of the beam shown in Fig. Take E = 200 kN/mm^2 and I = 200 × 10^6 mm^4 **[Ans.** y_B = 41.25 mm]



2. A beam of uniform cross section and flexural rigidity 50 MN-m^2 is hinged at *A* and rests on support *B*, 6 m from *A*. clockwise moment of 300 kN-m acts at *C*, 4 m from A. Determine the deflection at *C* and also the maximum deflection and its position.

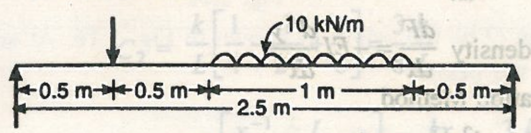
[Ans. y_C = 5.333 mm, $y_{max.}$ = 7.542 mm, at $x = 2\sqrt{2}$ m]

3. Determine the maximum deflection and its location in the beam shown below. The beam has a rectangular cross section 50 mm wide and 100 mm deep. Take E = 200 kN/mm^2.



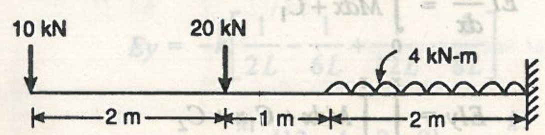
[Ans. y_{max} = 20.16 mm, at 1.84 m from *A*]

4. Determine the maximum deflection and its location in the beam shown below. The beam has a cross section 40 mm wide × 100 mm deep. Take E = 2 × 10^5 N/mm^2.



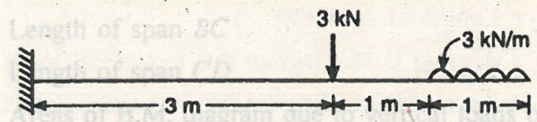
[Ans. y_{max} = 23.03 mm, at 1.85 m from *A*]

5. Find the slope and deflection at the free and of the cantilever shown below. Take E = 200 kN/mm^2, I = 40 × 10^6 mm^4.



[Ans. Slope = 5.042 × 10^{-3} rad, Deflection = 10.083 mm]

6. Find the slope and deflection at the free end of the cantilever shown below. Take EI = 10^{10} kN-mm^2.

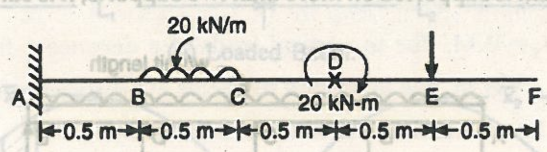


[Ans. Slope = 44 × 10^{-3} rad, deflection = 16.038 mm]

7. A beam of uniform cross section and flexural rigidity *EI*, length 3 L is hinged at one end and rests on a support 2 L from the hinge. There is a load *W* at the free end and a total Load *W* uniformly distributed over the length between L and 2 L from the hinge. Determine the deflection at the concentrated load point and also deflection at the mid-point of the supports.

$$\left[\textbf{Ans. } \frac{41WL^3}{48EI} \text{ downward, } \frac{7WL^3}{48EI} \text{ upward}\right]$$

8. Determine the slope and deflection at the force end of the cantilever shown below. Take E = 200 GPa, I = 200 × 10^6 mm^4.



[Ans. Slope = 1.573 × 10^{-3} rad., deflection = 2.85 mm]

6

Principle of Three Moments

6.1 INTRODUCTION

Before knowing the principle it is useful to discuss where it is going to be applied. Earlier, we have dealt those beam problems where the beam was supported on two supports. It was easy to determine the reactions at the support by using the normal equations of static equilibrium, since for two equations there were two unknowns.

When a beam is supported on more than two supports, it is called continuous. (Fig. 6.1)

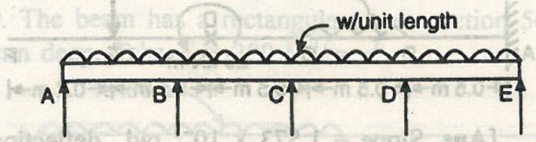


Fig. 6.1

If the moments over the intermediate supports of this continuous beam are known, then the Bending Moment Diagram can be drawn easily. The moments over the intermediate supports are determined by using the principle of three moments or the 'Clapeyron's theorem of three moments.'

The Clapeyron's theorem of three moments can be used to find the end support moment and draw the S.F. and B.M. diagrams, for any type of continuous beams. But we shall restrict our discussions only to the following types of continuous beams

(*i*) Continuous beams with simply supported ends

(*ii*) Continuous beams with fixed end supports.

(*iii*) Continuous beams with end span(s) overhanging.

6.2 CLAYPEYRON'S THEOREM OF THREE MOMENTS

If *BC* and *CD* are any two consecutive spans of a continuous beam subjected to an external loading, then the moments M_B, M_C and M_D at the supports *B*, *C* and *D* are given by

$$M_B L_1 + 2M_C(L_1 + L_2) + M_D L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2} \quad \text{....(1)}$$

where L_1 = Length of span *BC*

L_2 = Length of span *CD*

a_1 = Areas of B.M. diagram due to vertical loads on span *BC*

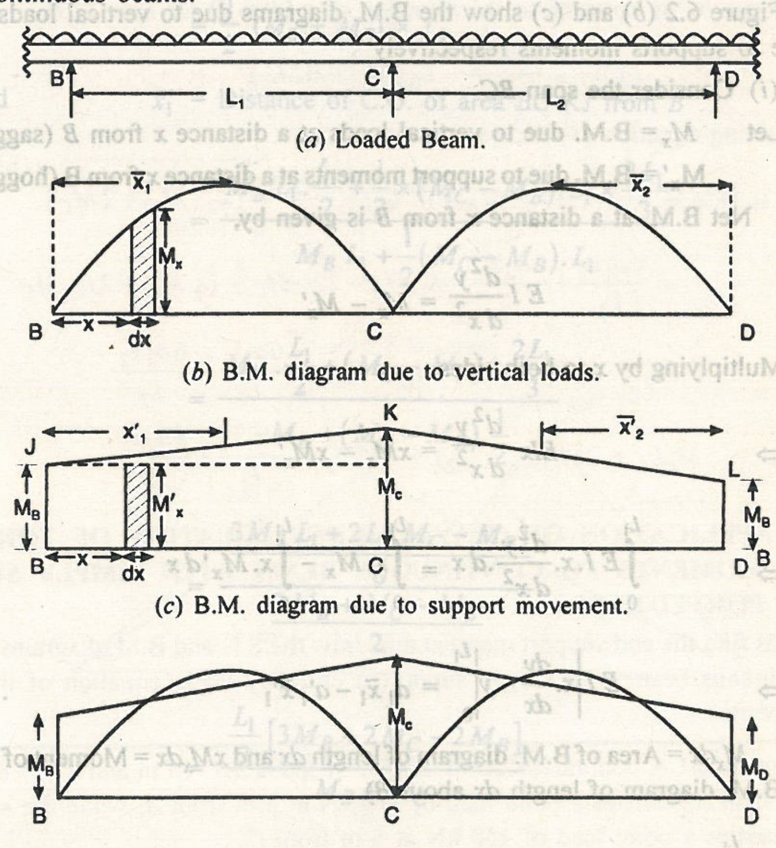
a_2 = Area of B.M. diagram due to vertical loads on span *CD*

$\bar{x}_1$ = Distance of C.G. of the B.M. diagram due to vertical loads on *BC* from *B*.

$\bar{x}_2$ = Distance of C.G. of the B.M. diagram due to vertical loads on *CD* from *D*.

The equation (1) is known as the equation of three moments or Claypeyron's equation.

Derivation : Figure 6.2 shows the length *BCD* (two consecutive spans) of a continuous beams.



(*a*) Loaded Beam.

(*b*) B.M. diagram due to vertical loads.

(*c*) B.M. diagram due to support movement.

(*d*) Resultant B.M. diagram.

Fig. 6.2

Let M_B, M_C and M_D are the support moments at B, C and D respectively.

Let L_1 = Length of span BC

L_2 = Length of span CD

a_1 = Area of B.M. diagram due to vertical loads on span BC

a_2 = Area of B.M. diagram due to vertical loads on span CD.

a'_1 = Area of B.M. diagram due to support moments M_B and M_C

a'_2 = Area of B.M. diagram due to support moments M_C and M_D

$\bar{x}_1$ = Distance of C.G. of B.M. diagram due to vertical loads on BC

$\bar{x}_2$ = Distance of C.G. of B.M. diagram due to vertical loads on CD

$\bar{x}'_1$ = Distance of C.G. of B.M. diagram due to support moments on BC

$\bar{x}'_2$ = Distance of C.G. of B.M. diagram due to support moments on CD

Figure 6.2 (*b*) and (*c*) show the B.M. diagrams due to vertical loads and due to supports moments respectively

(*i*) Consider the span BC

Let M_x = B.M. due to vertical loads at a distance x from B (sagging)

M_x' = B.M. due to support moments at a distance x from B (hogging)

Net B.M. at a distance x from B is given by,

$$EI\frac{d^2y}{dx^2} = M_x - M_x'$$

Multiplying by x to both sides

$$\Rightarrow \quad EIx\frac{d^2y}{dx^2} = xM_x - xM_x'$$

$$\Rightarrow \quad \int_0^{L_1} EI.x.\frac{d^2y}{dx^2}.dx = \int_0^{L_1} x.M_x - \int_0^{L_1} x.M_x'dx$$

$$\Rightarrow \quad EI\left|x.\frac{dy}{dx} - y\right|_0^{L_1} = a_1\bar{x}_1 - a'_1\bar{x}'_1 \qquad ...(1)$$

($M_x dx$ = Area of B.M. diagram of length dx and $xM_x dx$ = Moment of area of B.M. diagram of length dx about B).

$$\int_0^{L_1} x.M_x dx = \bar{a}_1\bar{x}_1 \text{ and so on.}$$

Substituting the limits in L.H.S. of equation (1), we have

$$EI\left[\left\{L_1\left(\frac{dy}{dx}\right)_{\text{at } C} - y_C\right\} - \left\{0 \times \left(\frac{dy}{dx}\right)_{\text{at } B} - y_B\right\}\right]$$

$$= a_1\bar{x}_1 - a'_1\bar{x}_1$$

or $\quad EI\,[(L_1 i_C - y_C) - (0 - y_B)] = a_1\bar{x}_1 - a'_1\bar{x}_1$

But deflection at B and C are zero

hence $\quad y_B = 0$ and $y_C = 0$

Hence above equation becomes as

$$[EIL_1 \,.\, i_C] = a_1\bar{x}_1 - a'_1\bar{x}'_1 \qquad ...(2)$$

But $\quad a'_1$ = Area of B.M. diagram due to supports moments

= Area of trapezium BC KJ

$$= \frac{1}{2}(M_B + M_C) \times L_1$$

and $\quad \bar{x}'_1$ = Distance of C.G. of area BC KJ from B

$$= \frac{M_B L_1.\frac{L_1}{2} + \frac{1}{2} \times (M_C - M_B).L_1 \times \frac{2L_1}{3}}{M_B L_1 + \frac{1}{2}(M_C - M_B).L_1}$$

$$= \frac{M_B.\frac{L_1}{2} + (M_C - M_B) \times \frac{2L_1}{3}}{M_B + (M_C - M_B).\frac{1}{2}}$$

$$= \frac{\frac{3M_B L_1 + 2L_1(M_C - M_B)}{6}}{\frac{2M_B + M_C - M_B}{2}}$$

$$= \frac{\frac{L_1}{3}[3M_B + 2M_C - 2M_B]}{M_B + M_C}$$

$$= \left(\frac{M_B + 2M_C}{M_B + M_C}\right) \times \frac{L_1}{3}$$

Substituting the values of $\bar{a}_1'$ and $\bar{x}_1'$ in equation (2), we get

$$EIL_1 \cdot i_C = a_1\bar{x}_1 - \frac{1}{2}(M_B + M_C)L_1 \times \left(\frac{M_B+2M_C}{M_B+M_C}\right)\times\frac{L_1}{3}$$

$$= a_1\bar{x}_1 - \frac{L_1^2}{6}(M_B+2M_C)$$

or $$6EIi_C = \frac{6a_1\bar{x}_1}{L_1} - L_1(M_B+2M_C) \qquad ...(3)$$

(*ii*) Consider the span *CD*

Similarly considering the span *CD* and taking *D* as origin and *x* positive to the left, it can be shown that

$$6EI(-i_C) = \frac{6a_2\bar{x}_2}{L_2} - L_2(M_D+2M_C)$$

(we have put opposite sign with i_C because the direction of *x* from *B* for the span *BC* is opposite to the direction of *x* from *D* for span *CD*).

$$-6EIi_C = \frac{6a_2\bar{x}_2}{L_2} - L_2(M_D+2M_C) \qquad ...(4)$$

Adding equation (3) and (4), we get

$$0 = \frac{6a_1\bar{x}_1}{L_1} - L_1(M_B+2M_C) + \frac{6a_2\bar{x}_2}{L_2} - L_2(M_D+2M_C)$$

$$= \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2} - L_1M_B - 2L_1M_C - L_2M_D - 2L_2M_C$$

$$\Rightarrow \quad L_1M_B + L_2M_D + 2M_C(L_1+L_2) = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

$$\Rightarrow \quad M_BL_1 + 2M_C(L_1+L_2) + M_B L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

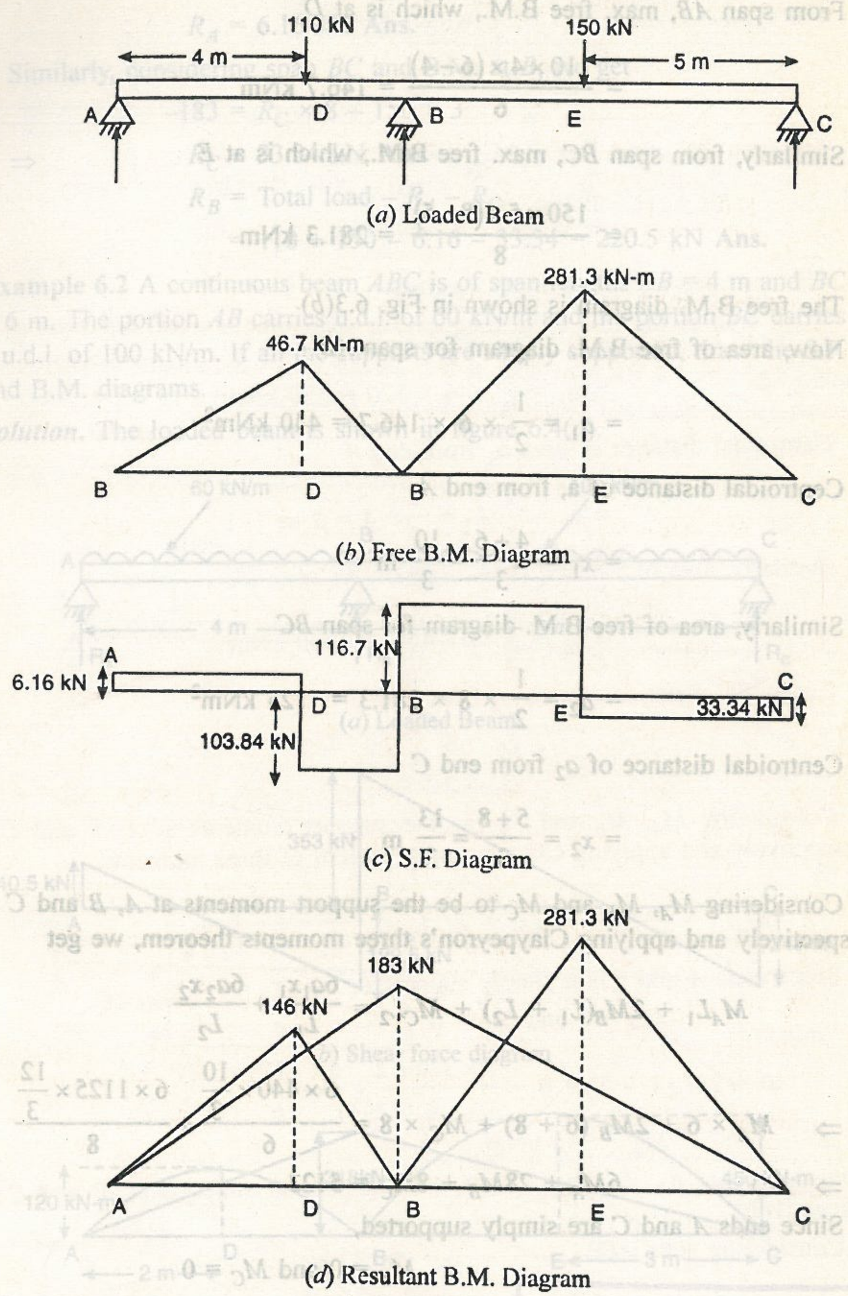
6.3 APPLICATION OF CLAYPEYRON'S EQUATION OF THREE MOMENTS TO CONTINUOUS BEAM WITH SIMPLY SUPPORTED ENDS

Let us find the end support moment and draw the S.F. and B.M. diagrams for continuous beam with simply supported ends, by using equation of three moments.

Example 6.1 A continuous beam *ABC* is of spans *AB* = 6 m and *BC* = 8 m. The span *AB* carries a point load of 110 kN at 4 m from *A*, while the span *BC* carries a point load of 150 kN at 5 m from *C*.

If supports *A*, *B* and *C* are simply supported, find the support moments and reactions. Draw the S.F. and B.M. diagrams.

Solution.



(*a*) Loaded Beam

(*b*) Free B.M. Diagram

(*c*) S.F. Diagram

(*d*) Resultant B.M. Diagram

Fig. 6.3

Given Length of span *AB*, L_1 = 6 m.

Length of span *BC*, L_2 = 8 m

The free B.M. diagrams for both the spans are triangular.

From span AB, max. free B.M., which is at D

$$= \frac{110 \times 4 \times (6-4)}{6} = 146.7 \text{ kNm}$$

Similarly, from span BC, max. free B.M., which is at E

$$= \frac{150 \times 5 \times (8-5)}{8} = 281.3 \text{ kNm}$$

The free B.M. diagram is shown in Fig. 6.3(*b*)

Now, area of free B.M. diagram for span AB

$$= a_1 = \frac{1}{2} \times 6 \times 146.7 = 440 \text{ kNm}^2$$

Centroidal distance of a_1 from end A

$$= x_1 = \frac{4+6}{3} = \frac{10}{3} \text{ m}$$

Similarly, area of free B.M. diagram for span BC

$$= a_2 = \frac{1}{2} \times 8 \times 281.3 = 1125 \text{ kNm}^2$$

Centroidal distance of a_2 from end C

$$= x_2 = \frac{5+8}{3} = \frac{13}{3} \text{ m}$$

Considering M_A, M_B and M_C to be the support moments at A, B and C respectively and applying Claypeyron's three moments theorem, we get

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = \frac{6a_1x_1}{L_1} + \frac{6a_2x_2}{L_2}$$

$$\Rightarrow \quad M_A \times 6 + 2M_B\,(6+8) + M_C \times 8 = \frac{6 \times 440 \times \frac{10}{3}}{6} + \frac{6 \times 1125 \times \frac{12}{3}}{8}$$

$$\Rightarrow \quad 6M_A + 28M_B + 8M_C = 5123$$

Since ends A and C are simply supported,

$$M_A = 0 \text{ and } M_C = 0$$

$$\Rightarrow \quad 0 + 28M_B + 0 = 5123$$

$$\Rightarrow \quad M_B = 183 \text{ kNm } \textbf{Ans.}$$

Reactions. Considering R_A, R_B and R_C as the reactions at A, B and C respectively.

$$\text{B.M. at } B = -183 = R_A \times 6 - 110 \times (6-4)$$

(for hogging B.M. at support, sign is –ve)

$$R_A = 6.16 \text{ kN } \textbf{Ans.}$$

Similarly, considering span BC and B.M. at B, we get

$$-183 = R_C \times 8 - 150 \times 3$$

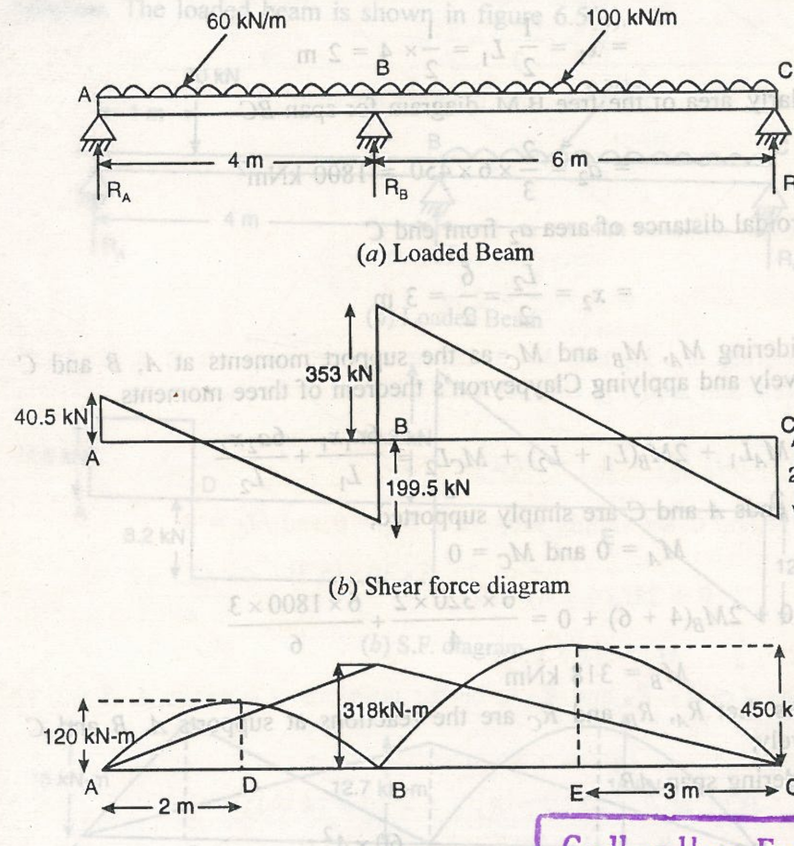
$$\Rightarrow \quad R_C = 33.34 \text{ kN } \textbf{Ans.}$$

$$R_B = \text{Total load} - R_A - R_C$$

$$= 110 + 150 - 6.16 - 33.34 = 220.5 \text{ kN } \textbf{Ans.}$$

Example 6.2 A continuous beam ABC is of span lengths $AB = 4$ m and $BC = 6$ m. The portion AB carries u.d.l. of 60 kN/m and the portion BC carries a u.d.l. of 100 kN/m. If all the supports are simply supported, draw the S.F. and B.M. diagrams.

Solution. The loaded beam is shown in figure 6.4(*a*).



(*a*) Loaded Beam

(*b*) Shear force diagram

(*c*) B.M. diagram

Fig. 6.4

Given length of span AB, $L_1 = 4$ m

length of span BC, $L_2 = 6$ m

For span AB, max. free B.M. which is at D

$$= \frac{60 \times 4^2}{8} = 120 \text{ kNm}$$

For span BC, max. free B.M. which is at E

$$= \frac{100 \times 6^2}{8} = 450 \text{ kNm}$$

First, we draw the free B.M. diagram.

Now, area of the free B.M. diagram for span AB

$$= a_1 = \frac{2}{3} \times 4 \times 120 = 320 \text{ kNm}^2$$

Centroidal distance of area a_1 from end A

$$= x_1 = \frac{1}{2} L_1 = \frac{1}{2} \times 4 = 2 \text{ m}$$

Similarly, area of the free B.M. diagram for span BC

$$= a_2 = \frac{2}{3} \times 6 \times 450 = 1800 \text{ kNm}^2$$

Centroidal distance of area a_2 from end C

$$= x_2 = \frac{L_2}{2} = \frac{6}{2} = 3 \text{ m}$$

Considering M_A, M_B and M_C as the support moments at A, B and C respectively and applying Claypeyron's theorem of three moments.

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = \frac{6a_1x_1}{L_1} + \frac{6a_2x_2}{L_2}$$

Since ends A and C are simply supported,

$$M_A = 0 \text{ and } M_C = 0$$

$$0 + 2M_B(4 + 6) + 0 = \frac{6 \times 320 \times 2}{4} + \frac{6 \times 1800 \times 3}{6}$$

$$\Rightarrow \quad M_B = 318 \text{ kNm}$$

Reactions. Let R_A, R_B and R_C are the reactions at supports A, B and C respectively,

Considering span AB,

$$\text{B.M. at } B = -318 = R_A \times 4 - \frac{60 \times 4^2}{2}$$

(For hogging moment at supports, B.M. is –ve)

$$R_A = \frac{1}{4}\left(\frac{60 \times 4^2}{2} - 318\right) = 40.5 \text{ kN}$$

Similarly considering span BC,

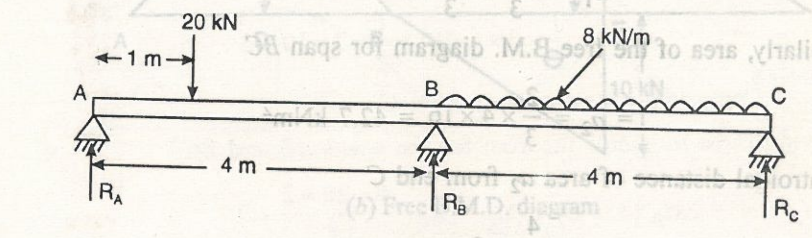
$$-318 = R_C \times 6 - 100 \times \frac{6^2}{2} \quad \Rightarrow R_C = 247 \text{ kN}$$

$$R_B = \text{Total Load} - R_A - R_C$$

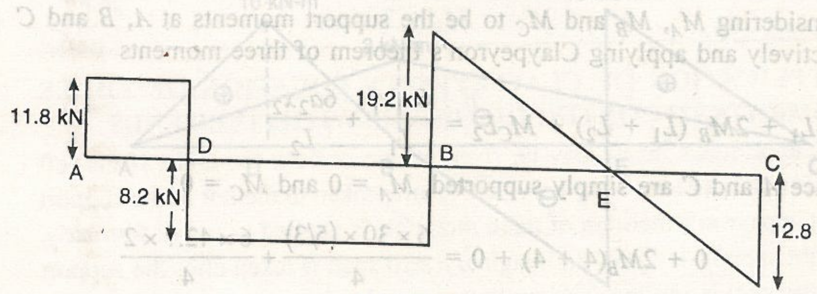
$$= 60 \times 4 + 100 \times 6 - 40.5 - 247 = 552.5 \text{ kN}$$

Example 6.3 Draw the S.F. and B.M. diagrams of a continuous beam ABC having span lengths $AB = 4$ m and $BC = 4$ m. The span AB is carrying a point load of 20 kN at a distance of 1 m from support A. The span BC carries a u.d.l. of intensity of 8 kN/m.

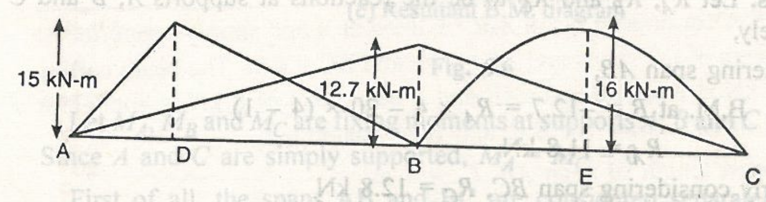
Solution. The loaded beam is shown in figure 6.5(a).



(a) Loaded Beam



(b) S.F. diagram



(c) B.M. diagram

Fig. 6.5

Length of span AB, $L_1 = 4$ m

Length of span BC, $L_2 = 4$ m

For span AB, Max. free B.M., which is at D

$$= \frac{20 \times 1 \times 3}{4} = 15 \text{ kNm}$$

For span BC, max. free B.M., which is at mid-span

$$= \frac{8 \times 4^2}{8} = 16 \text{ kNm}$$

Now, area of the free B.M. diagram for span AB

$$= a_1 = \frac{1}{2} \times 4 \times 15 = 30 \text{ kNm}^2$$

Centroidal distance of area a_1 from end A

$$= x_1 = \frac{1+4}{3} = \frac{5}{3} \text{ m.}$$

Similarly, area of the free B.M. diagram for span BC

$$= a_2 = \frac{2}{3} \times 4 \times 16 = 42.7 \text{ kNm}^2$$

Centroidal distance of area a_2 from end C

$$= x_2 = \frac{4}{2} = 2 \text{ m}$$

Considering M_A, M_B and M_C to be the support moments at A, B and C respectively and applying Claypeyron's theorem of three moments

$$M_A L_1 + 2M_B (L_1 + L_2) + M_C L_2 = \frac{6a_1 x_1}{L_1} + \frac{6a_2 x_2}{L_2}$$

Since A and C are simply supported, $M_A = 0$ and $M_C = 0$

$$0 + 2M_B(4 + 4) + 0 = \frac{6 \times 30 \times (5/3)}{4} + \frac{6 \times 42.7 \times 2}{4}$$

$$\Rightarrow \quad M_B = 12.7 \text{ kNm}$$

Reactions. Let R_A, R_B and R_C to be the reactions at supports A, B and C respectively,

Considering span AB,

$$\text{B.M. at } B = -12.7 = R_A \times 4 - 20 \times (4 - 1)$$

$$\Rightarrow \quad R_A = 11.8 \text{ kN}$$

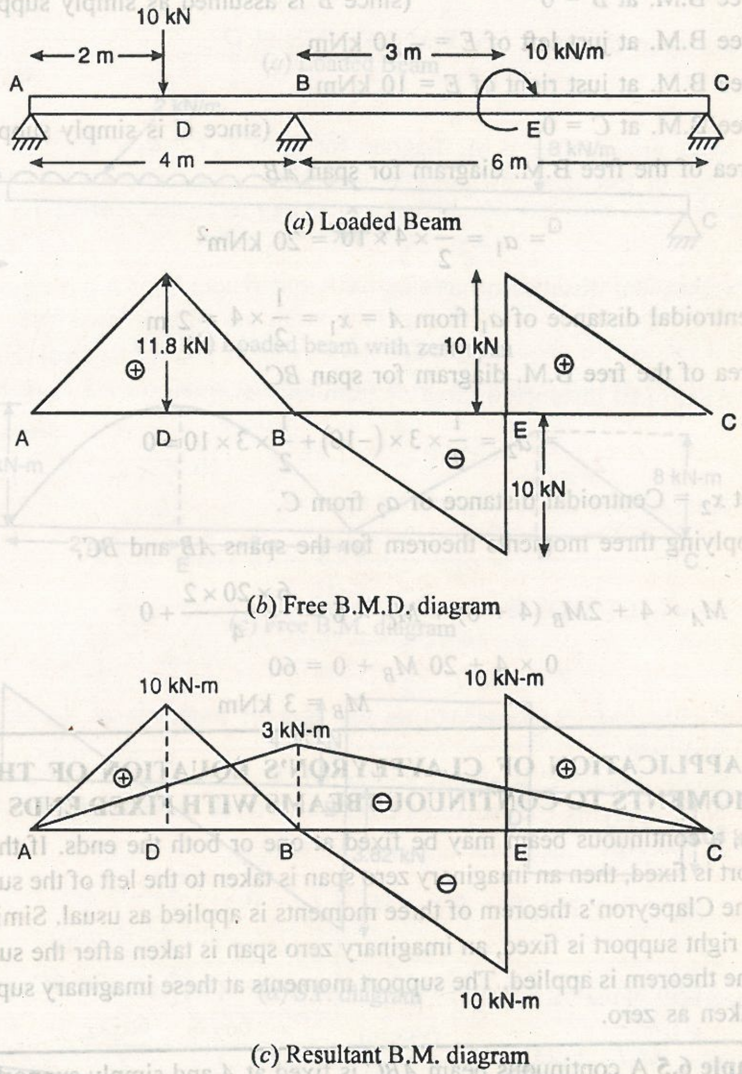
Similarly considering span BC, $R_C = 12.8$ kN

$$R_B = \text{Total load} - R_A - R_C$$

$$= 20 + 8 \times 4 - 11.8 - 12.8 = 27.4 \text{ kN}$$

Example 6.4 A continuous beam ABC is simply supported at A, B and C. It carries a central point load of 10 kN on the span AB and a central clockwise moment of 10 kNm at midspan BC. If $AB = 4$ m and $BC = 6$ m, draw the B.M. diagrams.

Solution.



(*a*) Loaded Beam

(*b*) Free B.M.D. diagram

(*c*) Resultant B.M. diagram

Fig. 6.6

Let M_A, M_B and M_C are fixing moments at supports A, B and C respectively. Since A and C are simply supported, $M_A = M_C = 0$

First of all, the spans AB and BC are considered separately as simply supported.

Free B.M. max. at D for span $AB = \frac{10 \times 4}{4} = 10$ kNm

The free B.M. max. for span AB is triangular with ordinate of 10 kNm at D.

for the span BC,

Free B.M. at $B = 0$ (since B is assumed as simply supported)

Free B.M. at just left of $E = -10$ kNm

Free B.M. at just right of $E = 10$ kNm

Free B.M. at $C = 0$ (since C is simply supported)

Area of the free B.M. diagram for span AB

$$= a_1 = \frac{1}{2} \times 4 \times 10 = 20 \text{ kNm}^2$$

Centroidal distance of a_1 from $A = x_1 = \frac{1}{2} \times 4 = 2$ m

Area of the free B.M. diagram for span BC

$$= a_2 = \frac{1}{2} \times 3 \times (-10) + \frac{1}{2} \times 3 \times 10 = 0$$

Let x_2 = Centroidal distance of a_2 from C.

Applying three moments theorem for the spans AB and BC,

$$M_A \times 4 + 2M_B (4 + 6) + M_C \times 6 = \frac{6 \times 20 \times 2}{4} + 0$$

or $$0 \times 4 + 20\, M_B + 0 = 60$$

or $$M_B = 3 \text{ kNm}$$

6.4 APPLICATION OF CLAYPEYRON'S EQUATION OF THREE MOMENTS TO CONTINUOUS BEAMS WITH FIXED ENDS

Often, a continuous beam may be fixed at one or both the ends. If the left support is fixed, then an imaginary zero span is taken to the left of the support and the Clapeyron's theorem of three moments is applied as usual. Similarly, if the right support is fixed, an imaginary zero span is taken after the support and the theorem is applied. The support moments at these imaginary supports are taken as zero.

Example 6.5 A continuous beam ABC is fixed at A and simply supported at B and C. Length of the spans are, $AB = 4$ m and $BC = 4$ m. The beam carries a u.d.l. of 2 kN/m over the span AB and a point load of 8 kN is applied at the midspan of BC. Draw the S.F. and B.M. diagram.

Solution. Given length of span AB, $L_1 = 4$ m

Length of span BC, $L_2 = 4$ m

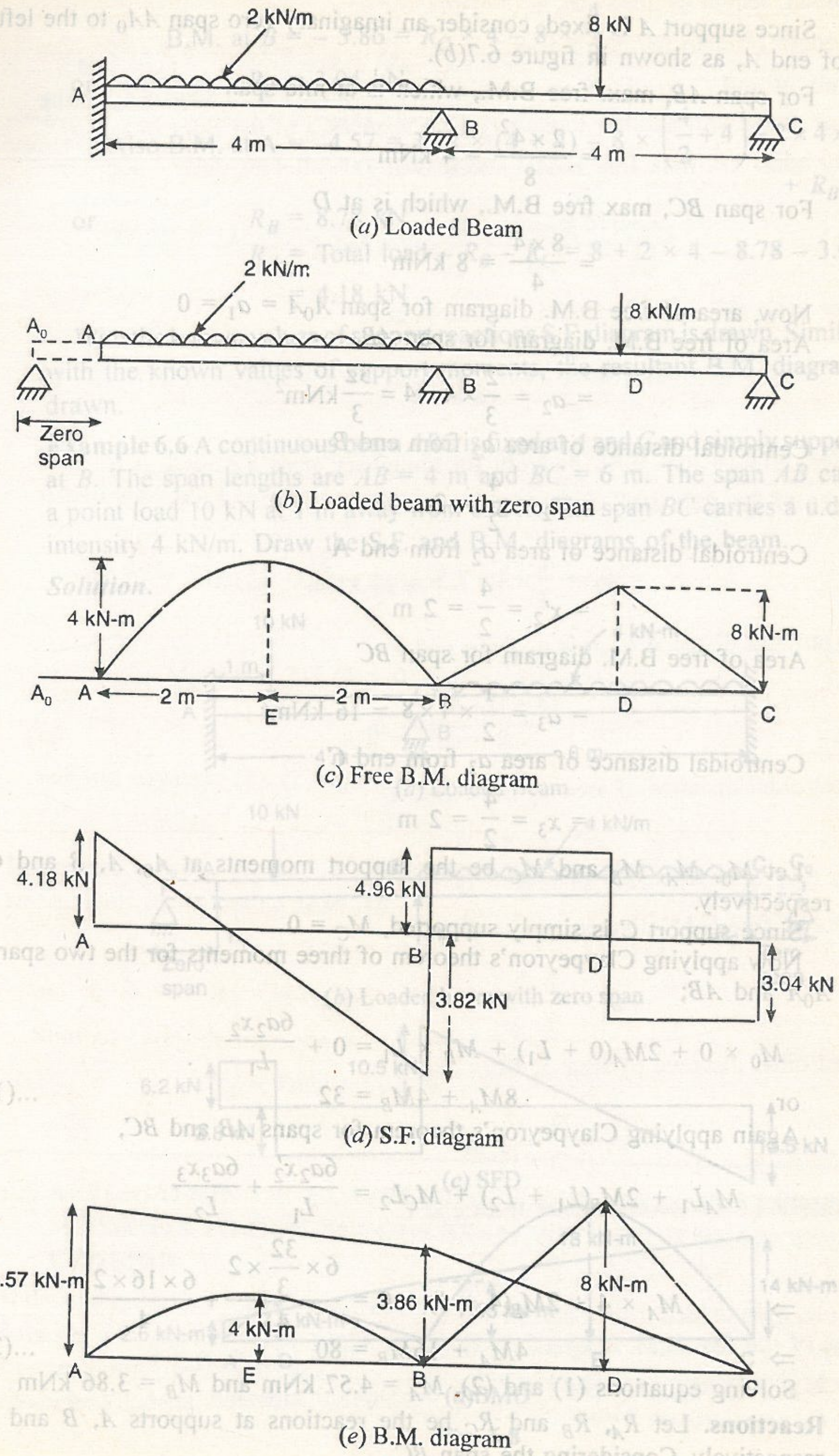


Fig. 6.7

Since support A is fixed, consider an imaginary zero span AA_0 to the left of end A, as shown in figure 6.7(b).

For span AB, max. free B.M., which is at mid-span

$$= \frac{2 \times 4^2}{8} = 4 \text{ kNm}$$

For span BC, max free B.M., which is at D

$$= \frac{8 \times 4}{4} = 8 \text{ kNm}$$

Now, area of free B.M. diagram for span $A_0A = a_1 = 0$

Area of free B.M. diagram for span AB

$$= a_2 = \frac{2}{3} \times 4 \times 4 = \frac{32}{3} \text{kNm}^2$$

Centroidal distance of area a_2 from end B

$$= x_2 = \frac{4}{2} = 2 \text{ m}$$

Centroidal distance of area a_2 from end A

$$= x'_2 = \frac{4}{2} = 2 \text{ m}$$

Area of free B.M. diagram for span BC

$$= a_3 = \frac{1}{2} \times 4 \times 8 = 16 \text{ kNm}^2$$

Centroidal distance of area a_3 from end C

$$= x_3 = \frac{4}{2} = 2 \text{ m}$$

Let M_0, M_A, M_B and M_C be the support moments at A_0, A, B and C respectively.

Since support C is simply supported, $M_C = 0$

Now applying Claypeyron's theorem of three moments for the two spans A_0A and AB;

$$M_0 \times 0 + 2M_A(0 + L_1) + M_B \times L_1 = 0 + \frac{6a_2x_2}{L_1}$$

or $$8M_A + 4M_B = 32 \quad ...(1)$$

Again applying Claypeyron's theorem for spans AB and BC,

$$M_AL_1 + 2M_B(L_1 + L_2) + M_CL_2 = \frac{6a_2x'_2}{L_1} + \frac{6a_3x_3}{L_2}$$

$$\Rightarrow \quad M_A \times 4 + 2M_B(4 + 4) + 0 = \frac{6 \times \frac{32}{3} \times 2}{4} + \frac{6 \times 16 \times 2}{4}$$

$$\Rightarrow \quad 4M_A + 16M_B = 80 \quad ...(2)$$

Solving equations (1) and (2), $M_A = 4.57$ kNm and $M_B = 3.86$ kNm

Reactions. Let R_A, R_B and R_C be the reactions at supports A, B and C respectively. Considering the span BC,

$$\text{B.M. at } B = -3.86 = R_C \times 4 - 8 \times \frac{4}{2}$$

or $$R_C = 3.04 \text{ kN}$$

$$\text{Also B.M. at A} = -4.57 = 3.04 \times (4 + 4) - 8 \times \left(\frac{4}{2} + 4\right) - 2 \times 4 \times \frac{4}{2} + R_B \times 4$$

or $$R_B = 8.78 \text{ kN}$$

$$R_A = \text{Total load} - R_B - R_C = 8 + 2 \times 4 - 8.78 - 3.04$$

$$= 4.18 \text{ kN}$$

With the known values of support reactions S.F. diagram is drawn. Similarly, with the known values of support moments, the resultant B.M. diagram is drawn.

Example 6.6 A continuous beam ABC is fixed at A and C and simply supported at B. The span lengths are $AB = 4$ m and $BC = 6$ m. The span AB carries a point load 10 kN at 1 m away from end A. The span BC carries a u.d.l. of intensity 4 kN/m. Draw the S.F. and B.M. diagrams of the beam.

Solution.

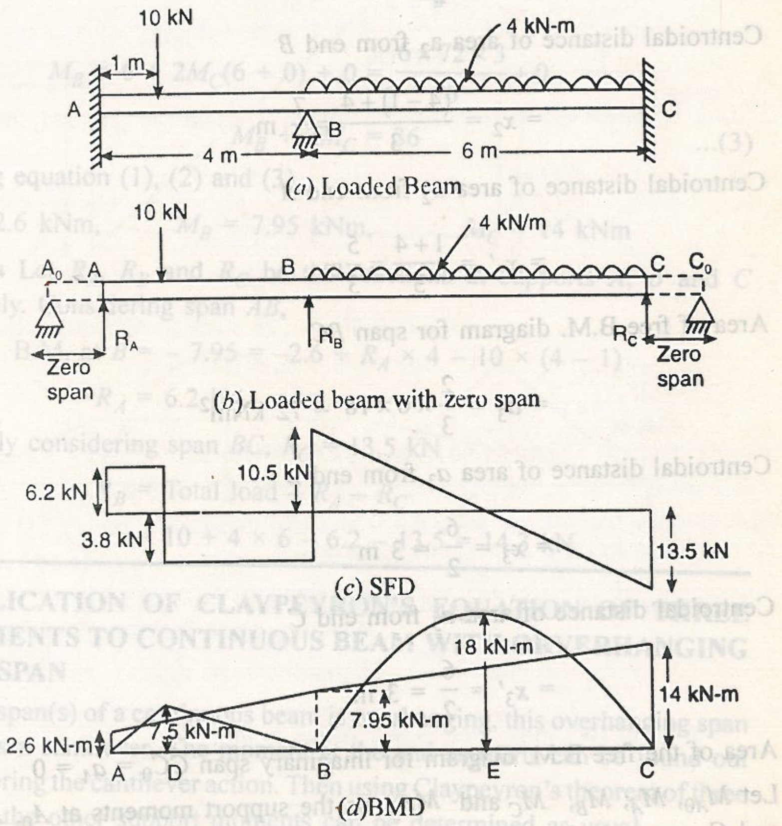


Fig. 6.8

Given, length of span AB, $L_1 = 4$ m

Length of span BC, $L_2 = 6$ m

Ends A and C are fixed, so, consider an imaginary zero span AA_0 to the left of end A and another imaginary zero span CC_0 to the right of end C,

For span AB, max free B.M., which is at D

$$= \frac{10 \times 1 \times (4-1)}{4} = 7.5 \text{ kNm}$$

For span BC, max free B.M., which is at E

$$= \frac{4 \times 6^2}{8} = 18 \text{ kNm}$$

Now, area of free B.M. diagram for imaginary span $AA_0 = a_1 = 0$

Area of the free B.M. diagram for span AB

$$= a_2 = \frac{1}{2} \times 4 \times 7.5 = 15 \text{ kNm}^2$$

Centroidal distance of area a_2 from end B

$$= x_2 = \frac{(4-1)+4}{3} = \frac{7}{3} \text{ m}$$

Centroidal distance of area a_2 from end A

$$= x_2' = \frac{1+4}{3} = \frac{5}{3} \text{ m}$$

Area of free B.M. diagram for span BC

$$= a_3 = \frac{2}{3} \times 6 \times 18 = 72 \text{ kNm}^2$$

Centroidal distance of area a_3 from end B

$$= x_3 = \frac{6}{2} = 3 \text{ m}$$

Centroidal distance of area a_3 from end C

$$= x_3' = \frac{6}{2} = 3 \text{ m}$$

Area of the free B.M. diagram for imaginary span $CC_0 = a_4 = 0$

Let M_{A0}, M_A, M_B, M_C and M_{C0} be the support moments at A_0, A, B, C and C_0 respectively.

Applying Claypeyron's theorem for spans A_0A and AB,

$$M_{A0} \times 0 + 2M_A(0 + L_1) + M_B \times L_1 = 0 + \frac{6a_2x_2}{L_1}$$

or $$8M_A + 4M_B = 52.5 \quad \text{...(1)}$$

Applying Claypeyron's theorem for spans AB and BC,

$$M_AL_1 + 2M_B(L_1 + L_2) + M_CL_2 = \frac{6a_2x_2'}{L_1} + \frac{6a_3x_3'}{L_2}$$

or $$M_A \times 4 + 2M_B(4 + 6) + M_C \times 6 = \frac{6 \times 15 \times \frac{5}{3}}{4} + \frac{6 \times 72 \times 3}{56}$$

or $$2M_A + 10M_B + 3M_C = 126.75 \quad \text{...(2)}$$

Again applying Claypeyron's theorem for spans BC and CC_0

$$M_BL_2 + 2M_C(L_2 + 0) + M_{C0} \times 0 = \frac{6a_3x_3}{L_2} + 0$$

$$\Rightarrow \quad M_B \times 6 + 2M_C(6 + 0) + 0 = \frac{6 \times 72 \times 3}{6} + 0$$

$$\Rightarrow \quad M_B + 2M_C = 36 \quad \text{...(3)}$$

Solving equation (1), (2) and (3)

$M_A = 2.6$ kNm, $\quad M_B = 7.95$ kNm, $\quad M_C = 14$ kNm

Reactions Let R_A, R_B and R_C be the reactions at supports A, B and C respectively. Considering span AB,

$$\text{B.M. at } B = -7.95 = -2.6 + R_A \times 4 - 10 \times (4 - 1)$$

or $$R_A = 6.2 \text{ kN}$$

Similarly considering span BC, $R_C = 13.5$ kN

$$R_B = \text{Total load} - R_A - R_C$$

$$= 10 + 4 \times 6 - 6.2 - 13.5 = 14.3 \text{ kN}$$

6.5 APPLICATION OF CLAYPEYRON'S EQUATION OF THREE MOMENTS TO CONTINUOUS BEAM WITH ORVERHANGING END SPAN

If the end span(s) of a continuous beam is overhanging, this overhanging span behaves as a cantilever. The moment at the end supports can be found out by considering the cantilever action. Then using Claypeyron's theorem of three moments, the other support moments can be determined as usual.

Example 6.7 Using Clapeyron's theorem of three moments, draw the S.F. and B.M. diagrams of the continuous beam *ABCD*, simply supported at *A*, *B* and *C* and the end *D* is free. The span lengths are, *AB* = 4 m, *BC* = 4 m and *CD* = 2 m. The span *AB* carries a point load of 5 kN at the midspan. The span BC carries a u.d.l. of 3 kN/m. The span *CD* carries another point load of 2 kN at the free end *D*.

Solution. Given, length of span *AB*, L_1 = 4m

Length of span *BC*, L_2 = 4 m.

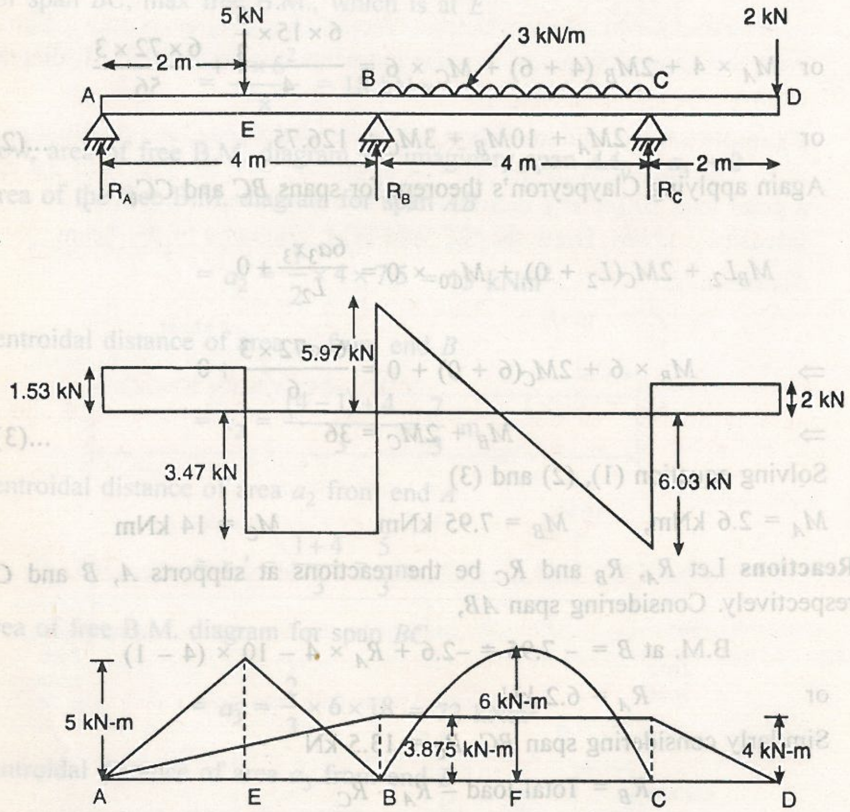


Fig. 6.9

For span *AB*, max. free B.M., which is at the mid span

$$= \frac{5 \times 4}{4} = 5 \text{ kNm}$$

For span BC, max. free B.M., which is at *F*

$$= \frac{3 \times 4^2}{8} = 6 \text{ kNm}$$

Now, area of free B.M. diagram for span *AB*

$$= a_1 = \frac{1}{2} \times 4 \times 5 = 10 \text{ kNm}^2$$

Centroidal distance of area a_1 from end *A*

$$= \frac{4}{2} = 2 \text{ m}$$

Area of the free B.M. diagram for span *BC*

$$= a_2 = \frac{2}{3} \times 4 \times 6 = 16 \text{ kNm}^2$$

Centroidal distance of area a_2 from end $B = \frac{4}{2} = 2$ m

Let M_A, M_B and M_C be the support moments at *A*, *B* and *C* respectively. Since end *A* is simply supported, $M_A = 0$

Due to cantilever action, $M_C = 2 \times 2 = 4$ kNm

Applying Claypeyron's theorem of three moments for spans *AB* and *BC*,

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = \frac{6a_1 x_1}{L_1} + \frac{6a_2 x_2}{L_2}$$

$$\Rightarrow \quad 0 \times L_1 + 2M_B(4 + 4) + 4 \times 4 = \frac{6 \times 10 \times 2}{4} + \frac{6 \times 16 \times 2}{4}$$

or $$M_B = 3.875 \text{ kNm}$$

Reactions. Let R_A, R_B and R_C be the reactions at supports *A*, *B* and *C* respectively.

Considering span *AB*,

B.M. at *B* $= -3.875 = R_A \times 4 - 5 \times 2$

$\Rightarrow \quad R_A = 4.53$ kN

Similarly considering span *BC*, $R_C = 8.03$ kN

$\therefore \quad R_B$ = Total load $- R_A - R_C$

$= 5 + 3 \times 4 + 2 - 1.53 - 8.03 = 9.44$ kN.

Example 6.8 A continuous beam *ABCDE* has its ends *A* and *E* free, and simply supported at *B*, *C* and *D*. The span lengths are *AB* = 1 m, *BC* = 4 m, *CD* = 6 m and *DE* = 1 m. Two point loads of 2 kN each are applied each at ends *A* and *E*. The span *BC* carries a central point load of 6 kN and the span *CD* carries a u.d.l. of 3 kN/m. Draw the S.F. and B.M. diagrams of the beam using Claypeyron's theorem of three moments.

Solution. Given, Length of overhung *AB*, L_1 = 1 m

Length of span *BC*, L_2 = 4 m

For span *BC*, max. free B.M., which is at *F*

$$= \frac{6 \times 4}{4} = 6 \text{ kNm}$$

For span CD, max. free B.M. which is at G

$$= \frac{3 \times 6^2}{8} = 13.5 \text{ kNm}$$

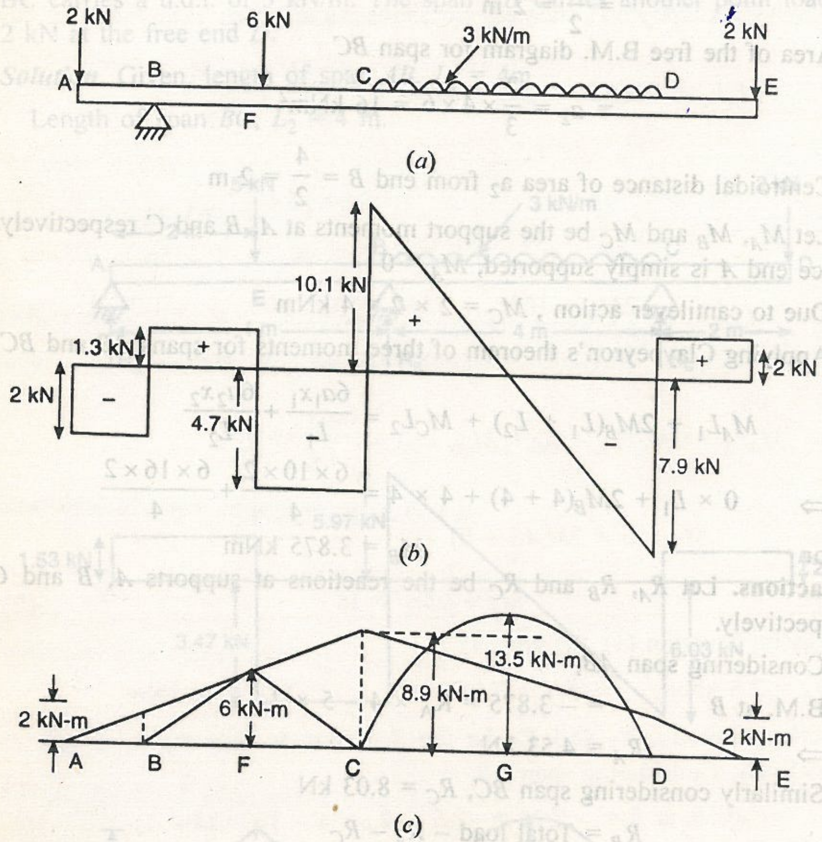


Fig. 6.10

Now, area of the free B.M. diagram for span BC

$$= a_1 = \frac{1}{2} \times 4 \times 6 = 12 \text{ kNm}^2$$

Centroidal distance of area a_1 from end B

$$= x_1 = \frac{4}{2} = 2 \text{ m}$$

Area of the free B.M. diagram for span BC

$$= a_2 = \frac{2}{3} \times 6 \times 13.5 = 54 \text{ kNm}^2$$

Centroidal distance of area a_2 from end D

$$= x_2 = \frac{6}{2} = 3 \text{ m}$$

Let M_B, M_C and M_D be the support moments at B, C and D respectively.

Considering cantilever action for span AB, $M_B = 2 \times 1 = 2$ kNm

Considering cantilever action for span DE, $M_D = 2 \times 1 = 2$ kNm

Applying Clapeyron's theorem of three moments for spans BC and CD,

$$M_B L_1 + 2M_C(L_1 + L_2) + M_C L_2 = \frac{6a_1x_1}{L_1} + \frac{6a_2x_2}{L_2}$$

or
$$2 \times 4 + 2M_C(4 + 6) + 2 \times 6 = \frac{6 \times 12 \times 2}{4} + \frac{6 \times 54 \times 3}{6}$$

or
$$M_C = 8.9 \text{ kNm}$$

Reactions. Let R_A, R_B and R_C be the reactions at supports A, B and C respectively.

Considering span BC

$$\text{B.M. at } C = -8.9 = -2 \times 5 + R_B \times 4 - 6 \times 2$$

$$\Rightarrow \quad R_B = 3.3 \text{ kN}$$

Similarly considering CD, $R_D = R_D = 9.9$ kN

$$R_C = \text{Total load} - R_B - R_D$$

$$= 2 + 6 + 3 \times 6 + 2 - 3.3 - 9.9 = 14.8 \text{ kN}.$$

SUMMARY

1. A beam supported on more than two supports is called a continuous beam. In this case the B.M. diagram is sagging (+) at mid-span but hogging (–) over the intermediate supports.
2. The B.M. diagram of a continuous beam may be drawn by superimposing the free B.M. diagram and the support moment diagram.
3. The support moments of a continuous beam may be found out by using the Clapeyron's theorem of three moments. The theorem states that, for any two consecutive spans AB and BC of a continuous beam,

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = \frac{6a_1x_1}{L_1} + \frac{6a_2x_2}{L_2}$$

where, M_A = Support moment at A
M_B = Support moment at B
M_C = Support moment at C
L_1 = Length of span AB
L_2 = Length of span BC

a_1 = Area of the free B.M. diagram for span AB

a_2 = Area of the free B.M. diagram for span BC

x_1 = Centroidal distance fro the free B.M. diagram for span AB from support A

x_2 = Centroidal distance of the free B.M. diagram for span BC from support C.

4. For a continuous beam with its ends simply supported, the moment at end supports are zero. The beam can be analysed by using this boundary condition and the Three Moments Theorem.
5. To apply the Three Moments Theorem for a continuous beam fixed at ends, an imaginary zero span should be introduced beyond the fixed ends.
6. For a continuous beam with overhanging ends, the moments at the end supports can be obtained by considering the cantilever action of the orverhanging portion. Then by applying Clapeyron's theorem of three moments, the beam can be easily analysed.

REVIEW QUESTIONS

Write short notes on the following :

(*i*) Principle of three moments

(*ii*) Claypeyron's Theorem

(*iii*) Continuous Beam.

NUMERICAL PROBLEMS

1. A continuous beam consists of three successive spans of 8 m, 10m and 6 m and carries loads of 6 kN/m and 8 kN/m respectively on the spans. Determine the bending moments and reactions at the supports.
   [Ans. (*i*) $M_A = M_B = 0$, $M_C = 32.2$ kN-m, $M_B = 40.16$ kNm,
   (*ii*) $R_A = 18.98$ kN, $R_B = 49.82$ kN, $R_C = 48.57$ kN, $R_D = 18.63$ kN]
2. Draw the S.F. and B.M. diagram of a continuous beam ABC of length 10 m which is fixed at A and is supported on B and C. The beam carries a u.d.l. of 2 kN/m length over the entire length. The spans AB and BC are equal to 5 m each.
   [Ans. (*i*) $M_A = 3.57$ kNm, $M_B = 5.357$ kNm, $M_C = 0$,
   (*ii*) $R_A = 5.357$ kN, $R_B = 8.571$ kN, $R_C = 6.071$]
3. A simply supported two span continuous beam ABC having span length $AB = BC = 3$ m carries a central point load of 10 kN at both the spans. Find the reactions and bending moments at the supports. Also draw the S.F. and B.M. diagrams. **(Ans.** $R_A = R_C = 5$ kN, $R_B = 10$ kN, $M_A = M_C = 0$, $M_B = 5.6$ kNm)

4. A continuous beam ABC is simply supported at A, B and C and having $AB = 6$ m, $BC = 4$ m. The span AB carries a point load of 3 kN at 2 *m* away from the support A. The span BC is carrying a u.d.l. of 1 kN/m. Find the reactions and bending moments at supports A, B and C. Also draw the S.F. and B.M. diagrams.
   (Ans. $R_A = 1.6$ kN, $R_B = 4$ kN, $R_C = 1.4$ kN, $M_B = -2.4$ kNm)
5. A continuous beam $ABCDE$ is simply supported at B, C, D and E. The span $AB = 1.5$ m, $BC = CD = DE = 3$ m. A point load of 20 kN is placed at free end A. The spans BC, CD and DE carry u.d.l. of intensities 40 kN/m, 30 kN/m and 50 kN/m respectively. Find the reactions and moments at the supports. Hence draw the S.F. and B.M. diagrams.
   (Ans. $R_B = 82.67$ kN, $R_C = 96.49$ kN, $R_D = 39.01$ kN,
   $R_E = 61.83$ kN, $M_B = -30$ kN-m, $M_C = -220$ kNm,
   $M_D = -39.5$ kNm, $M_E = 0$)

7

Redundant Frames

7.1 INTRODUCTION

What are frames ? A frame is an assemblage of a number of members, which resist geometrical distortion under any applied system of loading. Frames are used in the roofs of sheds at railway platforms, workshops, bridges and industrial buildings etc. For a number of given loading forces, the members of the frame are determined and then the members are designed to carry the required forces.

Before we discuss about redundant frames, let us know that frames are classified into

(*i*) Statically determinate frames and

(*ii*) Statically indeterminate frames.

Statically determinate frames are those frames which can be analysed with the help of equations of statics alone.

Redundant frames are statically indeterminant. As the name suggests, these frames have more members than it requires to be perfect. The frame which is composed of such members, which are just sufficient to keep the frame in equilibrium, when the frame is supporting an external load, is known as a perfect frame. The simplest perfect frame is a triangle which consists three members and three joints. The three members are AB, BC and AC where as the three joints are A, B and C. This frame can be easily analysed by the condition of equilibrium (Fig. 7.1).

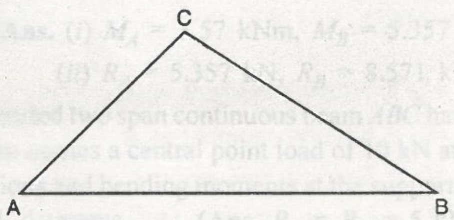


Fig. 7.1

Let the two members CD and BD and a joint D are added to the triangular frame ABC. Now, we get a frame $ABCD$ as shown in Fig. 7.2

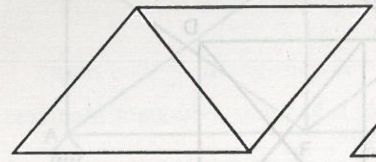


Fig. 7.2

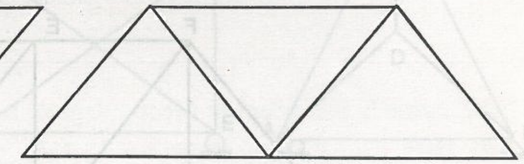


Fig 7.3

This frame can also be analysed by the conditions of equilibrium. This frame is also known as perfect frame.

Suppose we add a set of two members and a joint again, we get a perfect frame as shown in Fig. 7.3. Hence for a perfect frame, the number of joints and number of members are given by

$$n = 2j - 3$$

where n = number of members

j = number of joints

For Fig. 7.1

$n = 3, \quad j = 3$

$n = 2j - 3$

$\Rightarrow \quad 3 = 2 \times 3 - 3$

$\Rightarrow \quad 3 = 3$

Condition is satisfied.

For Fig. 7.2

$n = 5, \quad j = 4$

$\Rightarrow \quad 5 = 2 \times 4 - 3$

$\Rightarrow \quad 5 = 5$

Condition is satisfied.

For Fig. 7.3

$n = 7, \quad j = 5$

$n = 2j - 3$ gives

$\Rightarrow \quad 7 = 2 \times 5 - 3$

$\Rightarrow \quad 7 = 7$

Condition is satisfied.

When the members are less than that required by equation $n = 2j - 3$ then frame is called as imperfect frame. Such frame can not resist geometrical distortion under the action of loads.

7.2 REDUNDANT FRAMES

If the number of members are more than that required by equation $n = 2j - 3$, then such frames will be called as redundant frames.

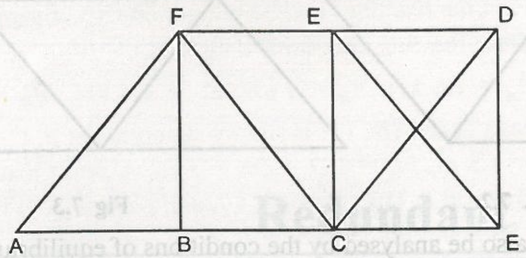


Fig. 7.4

Here $n = 12, \quad j = 7$

$n = 2j - 3$ gives

$12 = 2 \times 7 - 3$

$\Rightarrow \quad 12 = 11$

Condition is not satisfied.

Hence the frame is redundant.

7.3 DEGREE OF REDUNDANCY

The total degree of redundancy or indeterminancy of a frame is equal to the number by which the unknown reaction components exceed the conditon equations of equilibrium. The excess members are called as redundants.

Total degree of redundancy is given by

$$T = m - (2j - R)$$

where m = total number of members

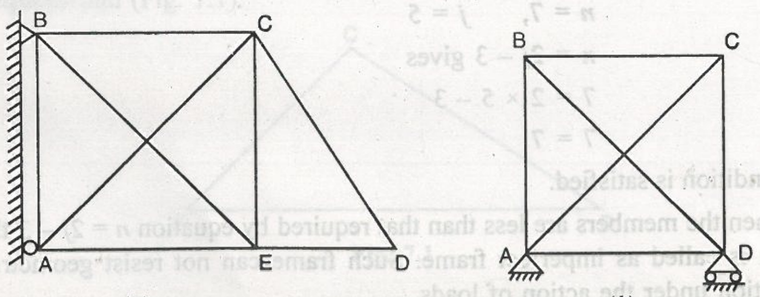
j = total number of joints

R = total number of reaction components

Reaction components are counted one for a roller, two for a hinge and three for a fixed support.

Example 7.1 Find the degree of redundancy of the frame shown in Fig. 7.5.

Solution.



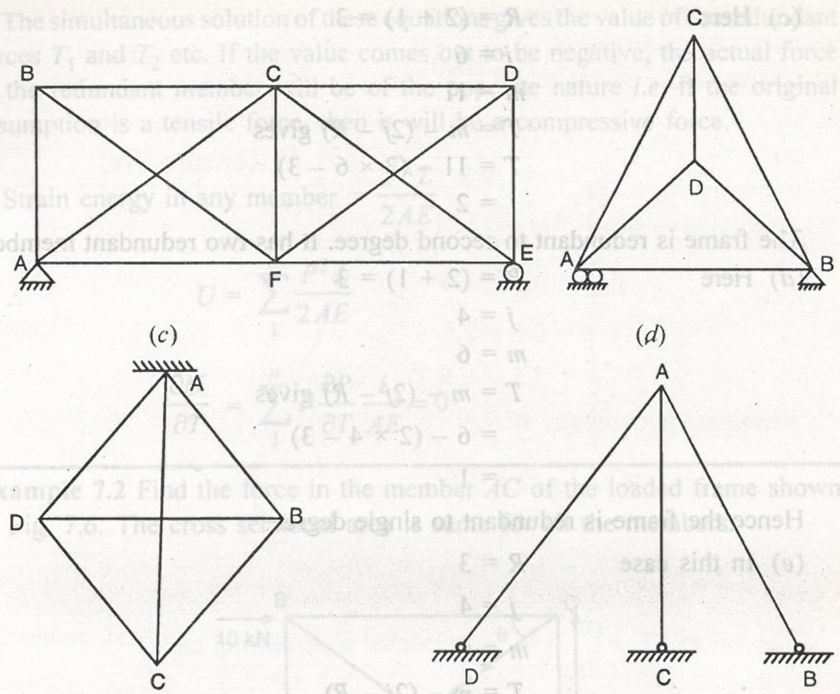


Fig. 7.5

(*a*) The total number of reaction components

$R = (2 + 1) = 3$

(Two for hinge support and one for roller).

Total number of joints = 5

(A, B, C, D and E are the 5 joints)

Total number of member = 8

(AB, BC, CD, DE, EA, BE, CE and AC are the different members)

Hence $T = m - (2j - R)$ gives

$T = 8 - (2 \times 5 - 3)$

$= 1$

Hence the frame is indeterminate to single degree.

(*b*) Here $R = (2 + 1) = 3$

$j = 4$

$m = 6$

$T = m - (2j - R)$

$\Rightarrow \quad T = 6 - (2 \times 4 - 3)$

$= 1$

The frame is redundant to single degree.

(*c*) Here $R = (2 + 1) = 3$

$j = 6$

$m = 11$

$T = m - (2j - R)$ gives

$T = 11 - (2 \times 6 - 3)$

$= 2$

The frame is redundant to second degree. It has two redundant members.

(*d*) Here $R = (2 + 1) = 3$

$j = 4$

$m = 6$

$T = m - (2j - R)$ gives

$= 6 - (2 \times 4 - 3)$

$= 1$

Hence the frame is redundant to single degree.

(*e*) In this case $R = 3$

$j = 4$

$m = 6$

$\therefore \quad T = m - (2j - R)$

$= 6 - (2 \times 4 - 3)$

$= 1$

Hence the frame is redundant to single degree.

(*f*) Here $R = 6$, as for stability of the frame there are three hinged supports.

$m = 3, \quad j = 4$

$\therefore \quad T = 3 - (2 \times 4 - 6) = 1$

Hence the frame is redundant to single degree.

7.4 ANALYSIS OF REDUNDANT FRAMES

To find the forces in the members of a loaded frame Castigliano's theorem of minimum strain energy is used.

If an elastic structure (frame) is subjected to forces and it is in a state of equilibrium, then the work stored is the smallest amount possible.

To use this method, the redundant members are replaced by the unknown forces (T_1, T_2, etc.) acting at the joints. Then Castigliano's theorem of minimum strain energy is applied to get

$$\frac{\partial U}{\partial T_1} = 0, \quad \frac{\partial U}{\partial T_2} = 0 \text{ etc.}$$

where U is the total strain energy (indusive of that in the redundant members) of the frame.

The simultaneous solution of these equations gives the value of the redundant forces T_1 and T_2 etc. If the value comes out to be negative, the actual force in the redundant member will be of the opposite nature *i.e.* if the original assumption is a tensile force, then it will be a compressive force.

$$\text{Strain energy in any member} = \frac{P^2 L}{2AE}$$

$$\therefore \quad U = \sum_{1}^{n} \frac{P^2 L}{2AE}$$

$$\frac{\partial U}{\partial T} = \sum_{1}^{n} P \frac{\partial P}{\partial T} \frac{L}{AE} = 0$$

Example 7.2 Find the force in the member AC of the loaded frame shown in Fig. 7.6. The cross sectional area is same for all the members.

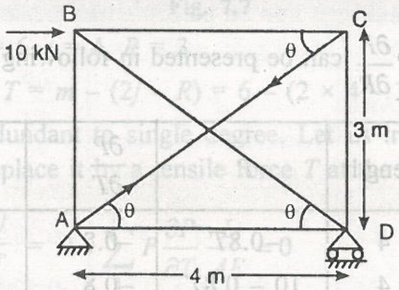


Fig. 7.6

Solution. The frame is redundant to single degree since

$$T = m - (2j - R) = 6 - (2 \times 4 - 3) = 1$$

Let us treat AC as the redundant member. Assuming that AC carries a tensile force T, we apply forces T at joints A and C and remove the member

$$\sin\theta = \frac{3}{5} = 0.6$$

$$\cos\theta = \frac{4}{5} = 0.8$$

Let us find the forces in various members

At C, resolving vertically,

$$P_{CD} = T\sin\theta = 0.6T \quad \text{(compressive)}$$

Resolving horizontally

$$P_{CB} = T\cos\theta = 0.8T \quad \text{(compressive)}$$

At A, resolving horizontally,

$$10 - P_{BC} = P_{BD}\cos\theta$$

$$\Rightarrow \quad P_{BD} = \frac{(10-0.8T)}{0.8} = (12.5 - T) \quad \text{(compressive)}$$

Resolving vertically

$$P_{BA} = P_{BD}\sin\theta$$

$$= 0.6\ (12.5 - \text{T})$$

$$= 7.5 - 0.6\ \text{T} \quad \text{(Tension)}$$

Resolving horizontally at D,

$$P_{AD} = P_{BD}\cos\theta$$

$$= (12.5 - T) \times 0.8$$

$$= 10 - 0.8\text{T}$$

The values of $P\dfrac{\partial P}{\partial R}$ can be presented in following table

Member	Length	P	$\dfrac{\partial P}{\partial T}$	$P\dfrac{\partial P}{\partial T}L$
BC	4	$-0.8T$	-0.8	$+2.56T$
AD	4	$10 - 0.8T$	-0.8	$-23 + 2.56T$
CD	3	$-0.6T$	-0.6	$+1.08T$
BA	3	$7.5 - 0.6T$	-0.6	$-13.5 + 1.08T$
CA	5	T	$+1.0$	$+5T$
BD	5	$T - 12.5T$	$+1.0$	$-62.5 + 5T$
				$-108 + 17.28T$

Since A and E are same, these terms are not used as they, being same for all the members cancel out

$$\therefore \quad \frac{\partial U}{\partial T} = 0 = \sum_{1}^{n} P\frac{\partial P}{\partial T}\frac{L}{AE} = 0$$

$$\Rightarrow \quad -108 + 17.28T = 0$$

$$\Rightarrow \quad T = +6.25 \text{ kN (Tension)}$$

Hence force in the member AC is 6.25 kN (Tensile).

Example 7.3 A loaded frame work is shown in Fig. 7.7. At point B, a load of 10 kN is applied horizontally. Find the force in the member DB. The cross sectional area of members BC, CA and AB is $2a$, while the members DC, DA and DB have area of cross section 'a'.

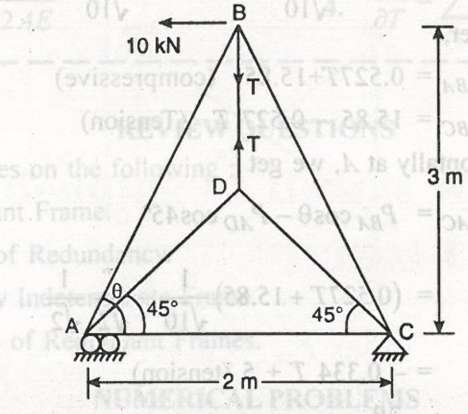


Fig. 7.7

Solution. Here $m = 6$, $j = 4$, $R = 3$

$$\therefore \quad T = m - (2j - R) = 6 - (2 \times 4 - 3) = 1$$

The frame is redundant to single degree. Let us treat BD as redundant member. We can replace it by a tensile force T at the joints B and D.

$$\frac{\partial U}{\partial T} = 0 = \sum_{1}^{n} P\frac{\partial P}{\partial T}\frac{L}{AE} = 0$$

Since E is same for all members

$$\therefore \quad \sum_{1}^{n} P\frac{\partial P}{\partial T}\frac{L}{A} = 0$$

Let us find the forces in various members.

At the joint D, resolving horizontally,

$$P_{AD}\cos 45° = P_{DC}\cos 45°$$

$$\Rightarrow \quad P_{AD} = P_{DC}$$

Resolving vertically,

$$P_{AD}\sin 45 + \text{PDC}\sin 45° = T$$

$$\Rightarrow \quad P_{AD} = P_{DC} = \frac{T}{\sqrt{2}} \quad \text{(Tension)}$$

At the joint B, resolving vertically,

$$P_{BA}\sin\theta = T + P_{BC}\sin\theta \qquad \text{...(1)}$$

Resolving horizontally,

$$P_{BA}\cos\theta + P_{BC}\cos\theta = 10 \quad ...(2)$$

Putting the values of $\sin\theta = \frac{3}{\sqrt{10}}$ and $\cos\theta = \frac{1}{\sqrt{10}}$ in equation (1) and (2) and solving together,

$$P_{BA} = 0.527T + 15.85 \quad \text{(compressive)}$$

$$P_{BC} = 15.85 - 0.527\,T \quad \text{(Tension)}$$

Resolving horizontally at A, we get

$$P_{AC} = P_{BA}\cos\theta - P_{AD}\cos 45°$$

$$= (0.527T + 15.85)\frac{1}{\sqrt{10}} - \frac{T}{\sqrt{2}}\frac{1}{\sqrt{2}}$$

$$= -0.334\,T + 5 \text{ (tension)}$$

The values of P and $\frac{\partial P}{\partial T}$ can be presented in the following table :

Member	Length	Area	P	$\frac{\partial P}{\partial T}$	$P\frac{\partial P}{\partial T}L$
BA	3.16	$2a$	$-(0.527T + 15.85)$	-0.527	$0.439T + 13.2$
AC	2.00	$2a$	$-0.334T + 5$	-0.334	$0.112T - 1.6T$
BC	3.16	$2a$	$15.85 - 0.527T$	-0.527	$-13.2 + 0.439T$
AD	$\sqrt{2}$	a	$+\frac{T}{\sqrt{2}}$	$+\frac{1}{\sqrt{2}}$	$+0.707T$
CD	$\sqrt{2}$	a	$+\frac{T}{\sqrt{2}}$	$+\frac{1}{\sqrt{2}}$	$+0.707T$
BD	2.00	a	$+T$	$+1$	$+2T$
				Total	$4.404T - 1.6T$

$$\therefore \quad \sum_{1}^{n} P\frac{\partial P}{\partial T}\frac{L}{A} = 0 = \frac{1}{a}(4.404T - 1.67)$$

$$\Rightarrow \quad T = 0.379 \text{ kN (Tension)}$$

Hence force in the member BD is 0.379 kN (Tension)

USEFUL RESULTS

1. $n = 2j - 3$
2. $T = m - (2j - R)$
3. $U = \sum_{1}^{n}\frac{P^2 L}{2AE}$
4. $\frac{\partial U}{\partial T} = \sum_{1}^{n} P\frac{\partial P}{\partial T}\frac{L}{AE} = 0$

REVIEW QUESTIONS

Write short notes on the following :

(*i*) Redundant Frame.

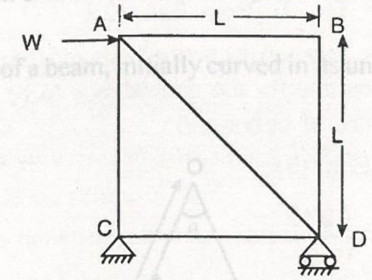
(*ii*) Degree of Redundancy.

(*iii*) Statically Indeterminate Frame.

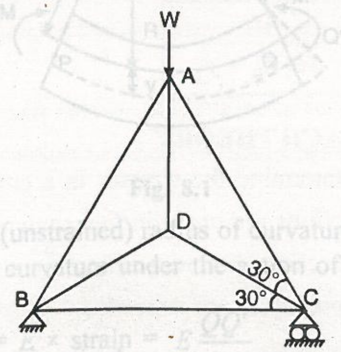
(*iv*) Analysis of Redundant Frames.

NUMERICAL PROBLEMS

1. The material and cross sectional area of the bars of the frame shown in Figure below are same. Show that force in AD is 0.707 W tensile.

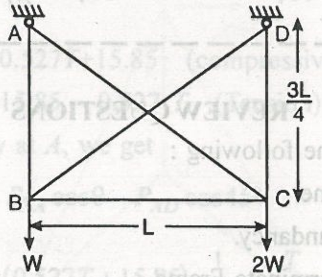


2. Determine the forces in the members of the frame work shown in figure below. The quantity AE is constant for all the members.



(**Ans.** P_{AC} = + 2.5 mN, P_{BC} = + 3.33 kN, P_{CD} = + 5.83 kN)

3. Determine the forces in the members of the frame shown in figure below, which is pinned to supports A and D and carries loads of W and $2W$ at B and C respectively. The members AB and CD are '$3a$' and the remainder have 'a' cross sectional area.



[**Ans.** $P_{AB} = P_{AC} = -\ 0.535$ W, $P_{BC} = +\ 0.33$ W,
$P_{BD} = P_{AD} = P_{CD} = -\ 0.07$ W)

8

Bending of Beams with Large Initial Curvature

8.1 INTRODUCTION

The result of simple bending *i.e.* $\frac{M}{I} = \frac{\tau}{y} = \frac{E}{R}$, can be applied to the beams having small initial curvature, but for curved beams for which the radius of curvature is more than 5 times the beam depth, this straight beam formula is not applicable.

Consider a portion of a beam, initially curved in its unstrained state as shown in Fig. 8.1.

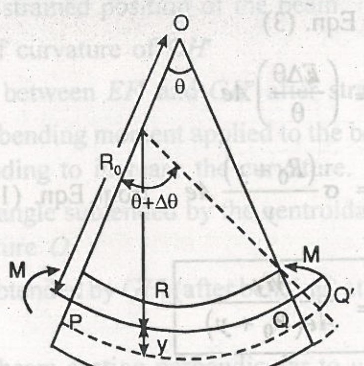


Fig. 8.1

Let R_0 be the initial (unstrained) radius of curvature of the neutral surface and R is the radius of curvature under the action of a bending moment M.

$$\sigma = E \times \text{strain} = E\frac{QQ'}{PQ}$$

$$= \frac{Ey\Delta\theta}{(R_0 + y)\theta} \quad \text{...(1)}$$

where y is the distance from the neutral axis

Total normal force on cross section = 0 for pure bending.

$$\Rightarrow \quad \int \sigma dA = \frac{E\Delta\theta}{\theta}\int \frac{ydA}{R_0 + y} = 0 \quad \text{...(2)}$$

$$M = \int \sigma y dA$$

$$= \frac{E\Delta\theta}{\theta}\int \frac{y^2 dA}{R_0 + y} \quad \text{...(3)}$$

But
$$\int \frac{y^2 dA}{R_0 + y} = \int \frac{[y(y + R_0) - R_0 y]}{R_0 + y} dA$$

$$= \int y dA - R_0 \int \frac{ydA}{R_0 + y}$$

$$= Ae - 0 \text{ [from Eqn. (2)]}$$

where e is the distance between the neutral axis and the principal axis through the centroid (e being positive for the neutral axis to be on the same side of the centroid as the centre of curvature).

Putting the value in Eqn. (3)

$$M = \left(\frac{E\Delta\theta}{\theta}\right) Ae$$

$$= \sigma \frac{(R_0 + y)}{y} Ae \quad \text{from Eqn. (1)}$$

$$\Rightarrow \quad \boxed{\sigma = \frac{My}{Ae(R_0 + y)}} \quad \text{...(4)}$$

8.2 THE WINKLER-BACH THEORY

This theory is used to determine the stresses in a curved beam.

The following assumptions are made in this analysis :

(*i*) Plane transverse sections before bending remain plane after bending.

(*ii*) Limit of proportionality is not exceeded.

(*iii*) Radial strain is negligible.

(*iv*) The material considered is isotropic and obeys Hooke's law.

Consider a portion of a beam $ABCD$ initially curved in its unstrained state as shown in Fig. 8.2

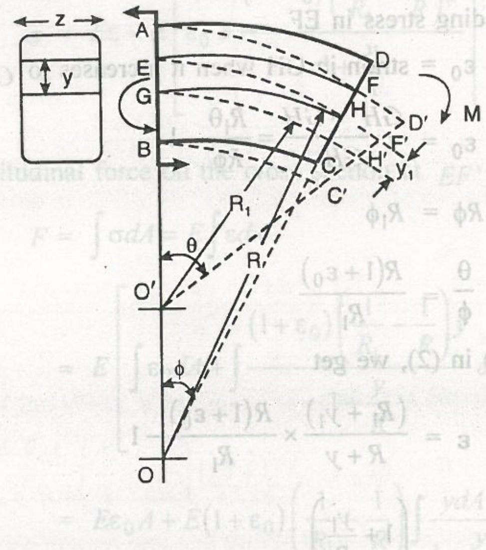


Fig. 8.2

Let, R = Radius of curvature of the centroidal axis GH

y = Distance of a fiber EF from GH

Let $ABC'D'$ be the strained position of the beam.

R_1 = Radius of curvature of GH'

y_1 = Distance between EF' and GH' after straining

M = Uniform bending moment applied to the beam, assumed positive when tending to increase the curvature.

ϕ = Original angle subtended by the centroidal axis GH at its centre of curvature O.

θ = Angle subtended by GH' (after bending) at the centre of curvature O'.

Let breadth of the beam section perpendicular to y be z, and let A be the constant area of cross-section *i.e.* $\Sigma\delta A = \Sigma z dy$, where δA is an element of area.

Now $\quad GH = R\phi$

$$EF = (R + y)\phi$$

$$EF' = (R_1 + y_1)\theta$$

Circumferential strain in EF,

$$\varepsilon = \frac{EF' - EF}{EF} = \frac{(R_1 + y_1)\theta}{(R + y)\phi} - 1 = \frac{\sigma}{E} \quad ...(2)$$

where σ = bending stress in EF

Let ε_0 = strain in GH when it increases to GH'

$$\varepsilon_0 = \frac{GH' - GH}{GH} = \frac{R_1\theta}{R\phi} - 1$$

$$\therefore \quad (1+\varepsilon_0)R\phi = R_1\phi$$

or

$$\frac{\theta}{\phi} = \frac{R(1+\varepsilon_0)}{R_1} \quad ...(3)$$

Substituting (3) in (2), we get

$$\varepsilon = \frac{(R_1 + y_1)}{R + y} \times \frac{R(1+\varepsilon_0)}{R_1} - 1$$

$$= \frac{\left(1 + \frac{y_1}{R_1}\right)}{\left(1 + \frac{y}{R}\right)} \times (1+\varepsilon_0) - 1$$

According to assumption of Winkler-Back Theory radial strain is zero,

$$\therefore \quad y = y_1$$

$$\varepsilon = \frac{\left(1 + \frac{y}{R_1}\right)}{\left(1 + \frac{y}{R}\right)} \times (1+\varepsilon_0) - 1$$

$$= \frac{1 + \frac{y_1}{R_1} + \varepsilon_0 + \varepsilon_0\frac{y_1}{R_1} - 1 - \frac{y}{R}}{1 + \frac{y}{R}}$$

Adding and subtracting $\varepsilon_0 \frac{y}{R}$ in the numerator, we get

$$\varepsilon = \varepsilon_0 + \frac{(1+\varepsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)y}{1 + \frac{y}{R}} \quad ...(4)$$

The tensile stress in EF' becomes,

$$\sigma = E\varepsilon = E\left[\varepsilon_0 + \frac{(1+\varepsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)y}{1 + \frac{y}{R}}\right] \quad ...(5)$$

The total longitudinal force on the cross section at EF' becomes,

$$F = \int \sigma dA = E\int \varepsilon dA$$

$$= E\left[\int \varepsilon_0 dA + \int \frac{(1+\varepsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)y}{1 + \frac{y}{R}} dA\right]$$

$$= E\varepsilon_0 A + E(1+\varepsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\int \frac{ydA}{1 + \frac{y}{R}} \quad ...(6)$$

The resisting moment at an axis through the centroid is,

$$M = \int \sigma dAy = \int E\varepsilon ydA$$

$$= \int E\varepsilon_0 ydA + \int \frac{E(1+\varepsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)}{1 + \frac{y}{R}} y^2 dA$$

Now $\int ydA = 0$, since y is measured from an axis through the centroid.

$$\therefore \quad M = E(1+\varepsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\int \frac{y^2 dA}{1 + \frac{y}{R}} \quad ...(7)$$

Let

$$\int \frac{y^2 dA}{1 + \frac{y}{R}} = Ah^2 \quad ...(8)$$

where h^2 = a constant for the cross section of the beam

$$\therefore \quad M = E(1+\varepsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)Ah^2 \quad ...(9)$$

Now consider

$$\int \frac{ydA}{1+\frac{y}{R}} = \int \frac{RydA}{R+y}$$

$$= \int\left(y - \frac{y^2}{R+y}\right)dA = \int ydA - \int \frac{y^2dA}{R+y}$$

$$= 0 - \frac{1}{R}\int \frac{y^2dA}{1+\frac{y}{R}}$$

$$\therefore \quad \int \frac{ydA}{1+\frac{y}{R}} = -\frac{1}{R}\int \frac{y^2dA}{1+\frac{y}{R}} = -\frac{1}{R}Ah^2 \quad ...(10)$$

Hence equation (6) becomes

$$F = E\varepsilon_0 A - E(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{Ah^2}{R} \quad ...(11)$$

Since transverse plane sections before bending remain plane after bending, hence

$$F = 0$$

or

$$0 = E\varepsilon_0 A - E(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{Ah^2}{R}$$

$$\therefore \quad \varepsilon_0 = (1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{h^2}{R} \quad ...(12)$$

Also from equation (9), we have

$$(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)h^2 = \frac{M}{AE}$$

Substituting in Eqn. (12), we have

$$\varepsilon_0 = \frac{M}{EAR} \quad ...(13)$$

Thus

$$\sigma = E\left[\varepsilon_0 + \frac{(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)y}{1+\frac{y}{R}}\right]$$

$$= E\left[\frac{M}{EAR} + \frac{M}{EAh^2}\left(\frac{y}{1+y/R}\right)\right]$$

$$= \left[\frac{M}{AR} + \frac{M}{Ah^2}\left(\frac{y}{1+y/R}\right)\right]$$

$$= \frac{M}{AR}\left(1+\frac{R^2}{h^2}\left(\frac{y}{R+y}\right)\right) \quad ...(14)$$

On the other side of GH, y will be negative, and stress will be compressive

$$\sigma = \frac{M}{AR}\left[1-\frac{R^2}{h^2}\left(\frac{y}{R-y}\right)\right] \quad ...(15)$$

If the bending moment is applied in such a manner that it tends to decrease the curvature of the beam, then Eqn. (14) will give compressive stress and Eqn. (15) will give tensile stress.

8.3 POSITION OF NEUTRAL AXIS

Neutral axis of a beam is the axis at which the bending stress is zero.

$\therefore$ At the neutral axis $\sigma = 0$

$$\therefore \quad \frac{M}{AR}\left[1+\frac{R^2}{h^2}\left(\frac{y}{R+y}\right)\right] = 0$$

$$\Rightarrow \quad \frac{R^2}{h^2}\left(\frac{y}{R+y}\right) = -1$$

$$R^2y = -Rh^2 - h^2y$$

$$(R^2+h^2)y = -Rh^2$$

$$y = -\left(\frac{Rh^2}{R^2+h^2}\right) \quad ...(16)$$

Hence the neutral axis is located below the centroidal axis

8.4 VALUES OF h^2

Now $$h^2 = \frac{1}{A}\int \frac{y^2dA}{1+\frac{y}{R}}$$

or
$$h^2 = \frac{R}{A}\int \frac{y^2 dA}{R+y}$$

$$= \frac{R}{A}\left[\int y dA - \int R dA + \int \frac{R^2 dA}{y+R}\right]$$

$$= \frac{R}{A}\left[0 - RA + \int \frac{R^2 dA}{y+R}\right]$$

$$= \frac{R^3}{A}\int \frac{dA}{y+R} - R^2 \qquad \ldots(17)$$

8.5 BEAMS WITH RECTANGULAR CROSS-SECTION

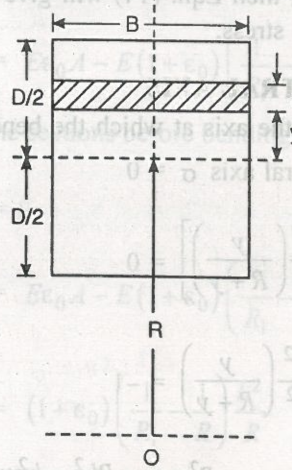


Fig. 8.3

Consider an elementary strip of width B and depth dy at a distance y from the centroid axis of a rectangular beam as shown in Fig. 8.3.

$$A = BD, \qquad dA = Bdy$$

$\therefore$
$$h^2 = \frac{R^3}{A}\int \frac{dA}{y+R} - R^2 \text{ gives}$$

or
$$h^2 = \frac{R^3}{BD}\int_{-D/2}^{+D/2} \frac{Bdy}{R+y} - R^2$$

or
$$h^2 = \frac{R^3}{BD}\left|\ln(R+y)\right|_{-D/2}^{+D/2} - R^2$$

or
$$h^2 = \frac{R^3}{BD}\ln\left(\frac{2R+D}{2R-D}\right) - R^2 \qquad \ldots(18)$$

8.6 BEAMS OF TRAPEZOIDAL CROSS SECTION

For the trapezoidal cross-section shown in Fig. 8.4, let $z = R + y$

$\therefore$
$$dz = dy$$

$$b = C + \left(\frac{B-C}{d_1+d_2}\right)(R_2 - z)$$

$$dA = bdy = \left[C + \left(\frac{B-C}{d_1+d_2}\right)(R_2 - z)\right]dz$$

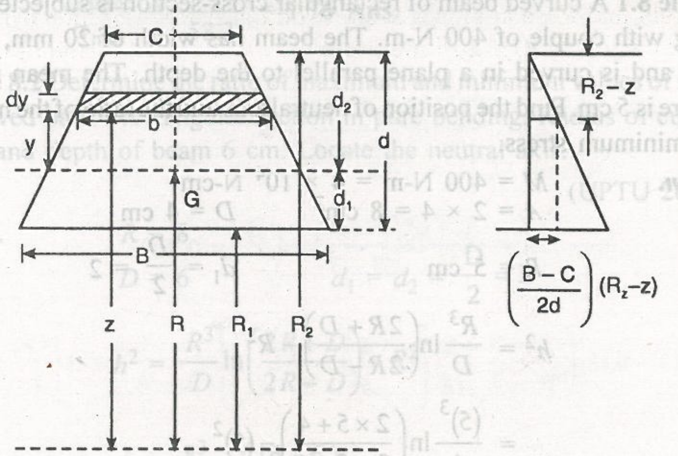


Fig. 8.4

$$h^2 = \frac{R^3}{A}\int_{R_1}^{R_2}\left[C + \frac{(B-C)}{d_1+d_2}(R_2 - z)\right]\frac{dz}{z} - R^2$$

$$= \frac{R^3}{A}\left[\int_{R_1}^{R_2}\frac{Cdz}{z} + \frac{(B-C)}{d_1+d_2}\int_{R_1}^{R_2}\left(\frac{R_2 - z}{z}\right)dz\right] - R^2$$

$$= \frac{R^3}{A}\left[C\left|\ln z\right|_{R_1}^{R_2} + \left(\frac{B-C}{d_1+d_2}\right)\left|R_2 \ln z - z\right|_{R_1}^{R_2}\right] - R^2$$

$$= \frac{R^3}{A}\left[C\ln\frac{R_2}{R_1} + \left(\frac{B-C}{d_1+d_2}\right)\left\{R_2\ln\frac{R_2}{R_1} - (R_2 - R_1)\right\}\right] - R^2$$

$$\boldsymbol{h^2 = \frac{R^3}{A}\left[C\ln\left(\frac{R+d_2}{R-d_1}\right) + \left(\frac{B-C}{d}\right)(R+d_2)\ln\left(\frac{R+d_2}{R-d_1}\right) - (B-C)\right] - R^2}$$

where $A = \left(\frac{B+C}{2}\right)d$

$$d_1 = \frac{d}{3}\left(\frac{B+2C}{B+C}\right)$$

$$d_2 = d - d_1$$

Example 8.1 A curved beam of rectangular cross-section is subjected to pure bending with couple of 400 N-m. The beam has width of 20 mm, depth of 40 mm and is curved in a plane parallel to the depth. The mean radius of curvature is 5 cm. Find the position of neutral axis and the ratio of the maximum to the minimum stress.

Solution. $M = 400$ N-m $= 4 \times 10^4$ N-cm

$A = 2 \times 4 = 8$ cm^2 $\qquad D = 4$ cm

$R = 5$ cm $\qquad d_1 = \frac{D}{2} = 2$

$$h^2 = \frac{R^3}{D}\ln\left(\frac{2R+D}{2R-D}\right) - R^2$$

$$= \frac{(5)^3}{4}\ln\left(\frac{2\times5+4}{2\times5-4}\right) - (5)^2$$

$$= 1.478$$

Location of neutral axis is given by

$$y = -\frac{Rh^2}{R^2+h^2}$$

$$= -\frac{5\times1.478}{25+1.478} = -0.279 \text{ cm } \textbf{Ans.}$$

(*i.e.* towards the centre of curvature)

Bending stress at the inside face will be maximum

$$\sigma_1 = -\frac{M}{AR}\left[1 - \frac{R^2}{h^2}\left(\frac{d_1}{R-d_1}\right)\right]$$

$$= -\frac{4\times10^4}{8\times5}\left[1 - \frac{25}{1.478}\left(\frac{2}{5-2}\right)\right]$$

$$= -10276.46 \text{ N/cm}^2 = 102.8 \text{ N/mm}^2 \text{ (compressive)}$$

Similarly, minimum bending stress will occur at the outside face.

$$\sigma_2 = -\frac{M}{AR}\left[1 + \frac{R^2}{h^2}\left(\frac{d_2}{R+d_2}\right)\right]$$

$$= -\frac{4\times10^4}{8\times5}\left[1 + \frac{25}{1.478}\left(\frac{2}{5+2}\right)\right]$$

$$= 5832.7 \text{ N/cm}^2$$

$$= 58.3 \text{ N/mm}^2 \text{ (tensile)}$$

$$\therefore \quad \frac{\sigma_{max}}{\sigma_{min}} = \frac{102.8}{58.3} = 1.76 \textbf{ Ans.}$$

Example 8.2 Determine the ratio of maximum and minimum values of stresses for a curved bar of rectangular section in pure bending. Radius of curvature is 8 cm and depth of beam 6 cm. Locate the neutral axis.

(UPTU 2001-02)

Solution. $R = 8$

$D = 6 \qquad d_1 = d_2 = \frac{D}{2} = 3$

$$h^2 = \frac{R^3}{D}\ln\left(\frac{2R+D}{2R-D}\right) - R^2$$

$$= \frac{8^3}{6}\ln\left(\frac{16+6}{16-6}\right) - 64 = 3.28$$

Location of neutral axis,

$$y = -\frac{Rh^2}{R^2+h^2} = -\frac{8\times3.28}{64+3.28} = 0.39 \text{ cm } \textbf{Ans.}$$

$$\sigma_{max} = -\frac{M}{AR}\left[1 - \frac{R^2}{h^2}\left(\frac{d_1}{R-d_1}\right)\right]$$

$$\sigma_{min} = -\frac{M}{AR}\left[1 + \frac{R^2}{h^2}\left(\frac{d_2}{R+d_2}\right)\right]$$

$$\therefore \quad \frac{\sigma_{max}}{\sigma_{min}} = \frac{1 - \frac{64}{3.28}\left(\frac{3}{8-3}\right)}{1 + \frac{64}{3.28}\left(\frac{3}{8+3}\right)} = \frac{-10.707}{6.32} = -1.69$$

–ve sign shows that σ_{max} is compressive

$$\therefore \quad \frac{\sigma_{max}}{\sigma_{min}} = 1.69 \ \textbf{Ans.}$$

Example 8.3 A curved bar of square section, 3 cm sides and mean radius of curvature 4.5 cm is initially unstressed. If a bending moment of 300 N-m is applied to the bar tending to straighten it, find the stresses at the inner and outer faces.

Solution. Given $R = 4.5$ $\quad d_1 = d_2 = D/2 = 1.5$ cm

$D = 3$ cm $\quad M = 3 \times 10^4$ N-cm

$A = 3 \times 3 = 9$ cm^2

$$h^2 = \frac{R^3}{D}\ln\left(\frac{2R+D}{2R-D}\right) - R^2$$

$$= \frac{4.5^3}{3}\ln\left(\frac{9+3}{9-3}\right) - 4.5^2 = 0.803$$

$$\sigma_{max} = \frac{-M}{AR}\left[1 - \frac{R^2}{h^2}\left(\frac{d_1}{R-d_1}\right)\right]$$

$$= \frac{3\times10^4}{9\times4.5}\left[1 - \frac{4.5^2}{0.803}\left(\frac{1.5}{4.5-1.5}\right)\right]$$

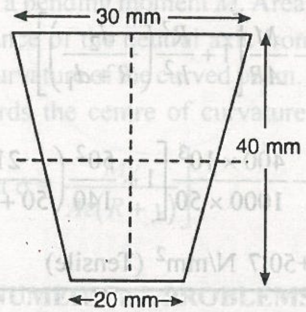
$= 8599$ N/cm^2 $= 86$ N/mm^2 (Tensile)

$$\sigma_{min} = \frac{M}{AR}\left[1 + \frac{R^2}{h^2}\left(\frac{d_2}{R+d_2}\right)\right]$$

$$= \frac{3\times10^4}{9\times4.5}\left[1 + \frac{(4.5)^2}{0.803}\left(\frac{1.5}{4.5+1.5}\right)\right]$$

$\sigma_1 = -5410$ N/cm^2 $= -54.10$ N/mm^2 (compressive)

Example 8.4 A curved beam, trapezoidal in cross section is subjected to pure bending with couple of 400 N-m. The mean radius of curvature is 50 mm. Find the position of the neutral axis and the ratio of the maximum to the minimum stress.



Solution. Given $\quad C = 20$ mm $\quad B = 30$ mm

$d = 40$ mm $\quad R = 50$ mm

$$d_1 = \frac{d}{3}\left(\frac{B+2C}{B+C}\right) = \frac{40}{3}\left(\frac{30+40}{30+20}\right) = 18.67 \text{ mm}$$

$$d_2 = d - d_1 = 40 - 18.67 = 21.33 \text{ mm}$$

$$A = \left(\frac{B+C}{2}\right)\alpha = \left(\frac{30+20}{2}\right)40 = 1000 \text{ mm}^2$$

$$h^2 = \frac{R^3}{A}\left[C\ln\left(\frac{R+d_2}{R-d_1}\right) + \left(\frac{B-C}{d}\right)(R+d_2)\ln\left(\frac{R+d_2}{R-d_1}\right) - (B-C)\right] - R^2$$

$$= \frac{50^3}{1000}\left[20\ln\left(\frac{50+21.33}{50-18.67}\right) + \left(\frac{30-20}{40}\right)(50+21.33)\ln\left(\frac{50+21.33}{50-18.67}\right) - (30-20) - 50^2\right]$$

$= 140$

Location of neutral axis,

$$y = -\frac{Rh^2}{R^2+h^2} = \frac{-50\times140}{2500+140} = -2.65 \text{ mm}$$

$$\sigma_{max} = -\frac{M}{AR}\left[1 - \frac{R^2}{h^2}\left(\frac{d_1}{R-d_1}\right)\right]$$

$$= \frac{400\times 10^3}{1000\times 50}\left[1-\frac{50^2}{140}\left(\frac{18.67}{50-18.67}\right)\right]$$

$$= -77.1 \text{ N/mm}^2 \text{ (compressive)}$$

$$\sigma_{min} = -\frac{M}{AR}\left[1+\frac{R^2}{h^2}\left(\frac{d_2}{R+d_1}\right)\right]$$

$$= \frac{400\times 10^3}{1000\times 50}\left[1+\frac{50^2}{140}\left(\frac{21.33}{50+21.33}\right)\right]$$

$$= +50.7 \text{ N/mm}^2 \text{ (Tensile)}$$

IMPORTANT RESULTS

1. $\sigma = \dfrac{M}{AR}\left[1+\dfrac{R^2}{h^2}\left(\dfrac{y}{R+y}\right)\right]$

2. At the neutral axis $\sigma = 0$, which gives $y = -\dfrac{Rh^2}{R^2+h^2}$

3. For rectangular section, $h^2 = \dfrac{R^3}{D}\ln\left(\dfrac{2R+D}{2R-D}\right) - R^2$

4. For trapezoidal section, $h^2 =$

$$\frac{R^3}{A}\left[C\ln\left(\frac{R+d_2}{R-d_1}\right)+\left(\frac{B-C}{d}\right)(R+d_2)\ln\left(\frac{R+d_2}{R-d_1}\right)-(B-C)\right]-R^2$$

where $A = \left(\dfrac{B+C}{2}\right)d,\quad d_1 = \dfrac{d}{3}\left(\dfrac{B+2C}{B+C}\right),\quad d_2 = d - d_1$

5. $\sigma = \dfrac{My}{Ae(R_0+y)}$

REVIEW QUESTIONS

1. Write short notes on the following :
   (*i*) Curved beam with large initial curvature
   (*ii*) Winkler Bach theory for curved beam
   (*iii*) Assumptions of the theory for curved beam.

2. Derive an expression for stress distribution in case of beam with large initial curvature.

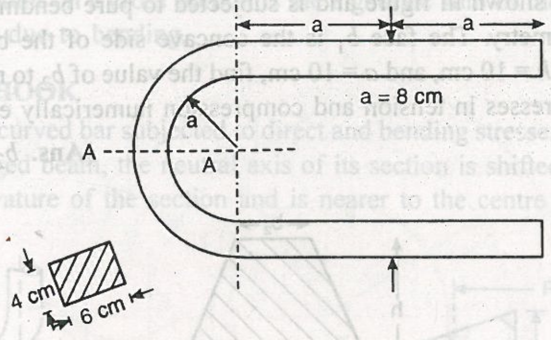
   [Hint : Derive Eqn (14) and (15)]

3. Find the stress σ at a distance y from the centroidal axis of curved beam subjected to a bending moment M. Area of cross-section of beam is A. e is the distance of the neutral axis from the centroidal axis and R is the radius of curvature of the curved beam. Assume that y is positive, if measured towards the centre of curvature of the beam.

   [Hint : Prove that $\sigma = \dfrac{My}{Ae(R+y)}$]

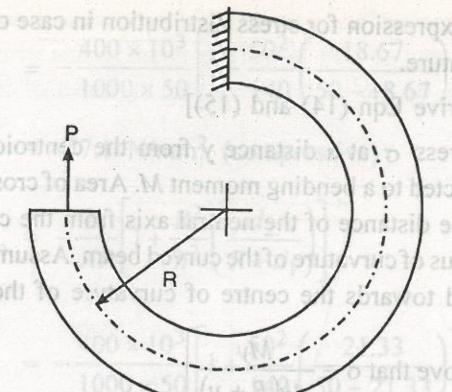
NUMERICAL PROBLEMS

1. Determine the maximum tensile and maximum compressive stresses across the section AA of the member loaded, as shown in figure. Load $P = 19620$ N. **(Ans.** 12642 kPa, 20482 kPa)



2. A bar of rectangular cross-section with a width of 60 mm and a thickness of 40 mm is bent in the shape of a horse shoe having a mean radius of 70 mm. Two equal and opposite forces of 10 kN each are applied at a distance of 12 cm from the centre line of the middle section. So that they tend to straighten the rod. find the maximum tensile and compressive stresses. **[Ans.** 74.69 MPa, – 33.4 MPa]

3. A curved beam shown in Fig. has a 30 mm square cross-section and a radius of curvature $R = 65$ mm. The beam is made of steel for which $E = 200$ GPa and $\nu = 0.30$. If $P = 6$ kN, determine the component of deflection of free end of other curved beam in the direction of P. **[Ans.** 1.107 mm]

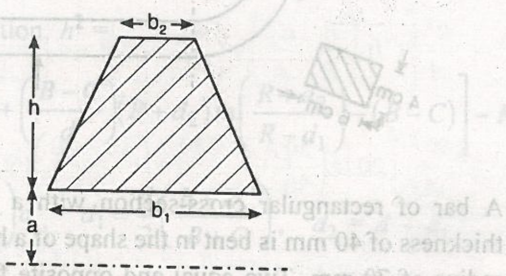


4. A curved bar of rectangular section 38 mm wide by 50 mm deep and of mean radius of curvature 100 mm is subjected to a bending moment of 1.5 kNm tending to straighten the bar. Find the position of the neutral axis and the magnitudes of the greatest bending stresses.

[**Ans.** e = 2.1 mm, 115.81 N/mm^2)

5. A curved beam with a circular centreline has the trapezoidal cross section shown in figure and is subjected to pure bending in its plane of symmetry. The face b_1 is the concave side of the beam. If b_1 = 10 cm, h = 10 cm, and a = 10 cm, find the value of b_2 to make extreme fibre stresses in tension and compression numerically equal.

[**Ans.** b_2 = 1.62 cm]



6. Determine the numerical value of the ratio $\sigma_{max}/\sigma_{min}$ for the case of pure bending of a curvature beam having a 2.5 cm × 2.5 cm square cross section if the radius of curvature of the centroidal axis is R = 3.75 cm.

[**Ans.** 1.59]

9

Stresses in Crane Hook, Circular Rings and Chain Links

9.1 INTRODUCTION

In real life the machine members subjected to bending are not always straight, before a bending moment is applied. Crane hooks, chain links and circular rings are such cases which have small radius of curvature. In all these cases the stress at any point on a cross section is the algebraic sum of the direct stress and the stress due to bending.

9.2 CRANE HOOK

The hook is a curved bar subjected to direct and bending stresses. Since crane hook is a curved beam, the neutral axis of its section is shifted towards the centre of curvature of the section and is nearer to the centre of curvature.

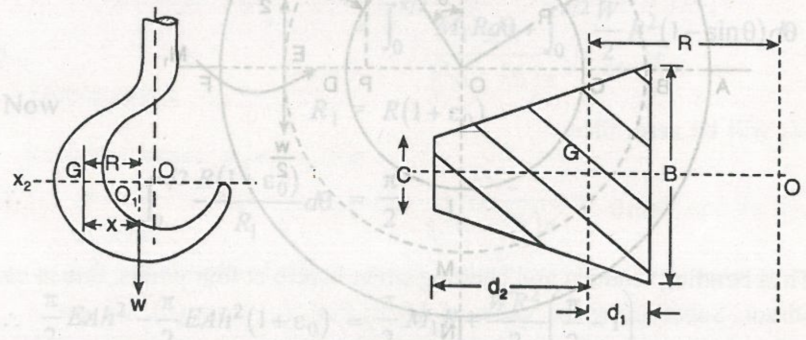


Fig. 9.1

For the crane hook shown in fig. 9.1, O is the centre of curvature and the load line passes through O_1. The radius of curvature of the centroid is R.

Bending moment about the centroid G is

$$M = Wx$$

This bending moment is such that it is tending to decrease the curvature, *i.e.* this is a negative bending moment. Therefore, bending stresses at point x_1 and x_2 are respectively,

$$\sigma_1 = \frac{Wx}{AR}\left\{\frac{R^2}{h^2}\left(\frac{d_1}{R-d_1}\right)-1\right\} \quad \text{(tensile)}$$

$$\sigma_2 = \frac{Wx}{AR}\left\{1+\frac{R^2}{h^2}\left(\frac{d_2}{R+d_2}\right)\right\} \quad \text{(compressive)}$$

$$\text{Direct stress } \sigma_d = \frac{W}{A} \quad \text{(tensile)}$$

$$\therefore \quad \text{Resultant stress at } x_1 = \sigma_1 + \sigma_d$$

9.3 STRESSES IN A RING

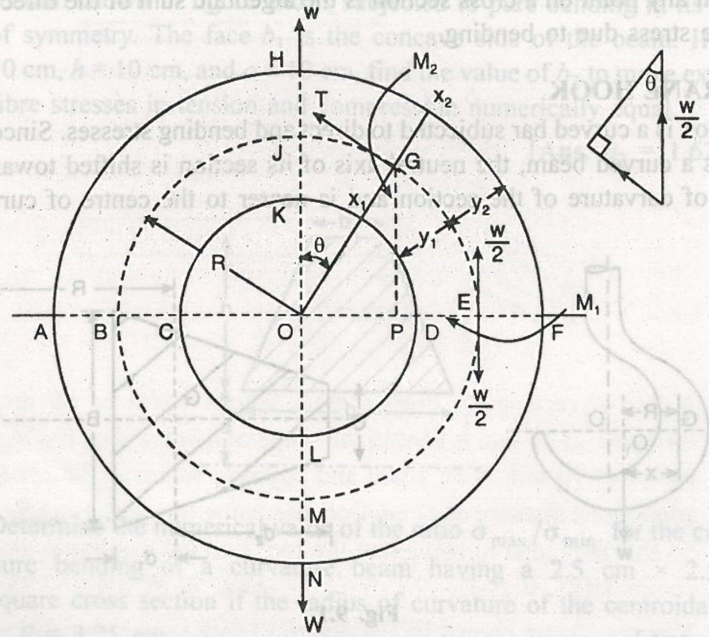


Fig. 9.2

Consider a circular ring loaded as shown in Fig. 9.2. Let M_2 be the bending moment at any section x_1-x_2 inclined at angle θ with the line of action of the

applied load W. The portion x_1DFx_2 of the rings is in equilibrium under the action of M_1 at DF, pull $W/2$ at DF and the moment M_2 at x_1-x_2 along with pull T at x_1-x_2.

we get,

$$M_2 = M_1 + \frac{W}{2}R(1-\sin\theta) \quad \text{...(a)}$$

Also

$$M_2 = E(1+\varepsilon_0)\times\left(\frac{1}{R_1}-\frac{1}{R}\right)Ah^2 \quad \text{...(b)}$$

Comparing Eqns. (*a*) and (*b*), we get,

$$E(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)Ah^2 = M_1 + \frac{W}{2}R(1-\sin\theta) \quad \text{...(c)}$$

Multiplying both sides by $Rd\theta$ and integrating from 0 to $\pi/2$, we have,

$$E\int_0^{\pi/2}(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)Ah^2 R d\theta$$

$$= \int_0^{\pi/2} M_1 R d\theta + \int_0^{\pi/2}\frac{W}{2}R^2(1-\sin\theta)d\theta$$

or

$$E\int_0^{\pi/2}\frac{R(1+\varepsilon_0)}{R_1}Ah^2 d\theta - E\int_0^{\pi/2}(1+\varepsilon_0)Ah^2 d\theta$$

$$= \int_0^{\pi/2} M_1 R d\theta + \int_0^{\pi/2}\frac{W}{2}R^2(1-\sin\theta)d\theta$$

Now

$$R_1 = R(1+\varepsilon_0)$$

$$\therefore \quad \int_0^{\pi/2}\frac{R(1+\varepsilon_0)}{R_1}d\theta = \frac{\pi}{2}$$

$$\therefore \quad \frac{\pi}{2}EAh^2 - \frac{\pi}{2}EAh^2(1+\varepsilon_0) = \frac{\pi}{2}M_1R + \frac{WR^2}{2}\left(\frac{\pi}{2}-1\right)$$

or

$$-\frac{\pi}{2}EAh^2\varepsilon_0 = \frac{\pi}{2}M_1R + \frac{WR^2}{2}\left(\frac{\pi}{2}-1\right) \quad \text{...(d)}$$

Now

$$T = \frac{W}{2}\sin\theta \quad \text{...(e)}$$

$$\therefore \quad EA\left[\varepsilon_0-(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{h^2}{R}\right] = \frac{W}{2}\sin\theta \qquad \text{...}(f)$$

From Eqn. (*c*), we have

$$E(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{Ah^2}{R} = \frac{M_1}{R}+\frac{W}{2}(1-\sin\theta)$$

Substituting in Eqn. (*f*), we get

$$EA\varepsilon_0 = \frac{W}{2}\sin\theta+\frac{M_1}{R}+\frac{W}{2}(1-\sin\theta)=\frac{W}{2}+\frac{M_1}{R}$$

$$\therefore \quad \varepsilon_0 = \frac{W}{2EA}+\frac{M_1}{EAR} \qquad \text{...}(g)$$

Putting in Eqn. (*d*), we get

$$M_1 = \frac{WR}{2}\left[\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)-1\right] \qquad \text{...}(h)$$

Substituting in Eqn. (*a*), we get

$$M_2 = \frac{WR}{2}\left[\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)-\sin\theta\right] \qquad \text{...}(i)$$

M_2 will be maximum at $\theta = 0°$ and $180°$

$$\therefore \quad M_{max} = \frac{WR^3}{\pi(R^2+h^2)} \qquad \text{...}(j)$$

M_2 will be zero, then

$$\sin\theta = \frac{2R^2}{\pi(R^2+h^2)} \qquad \text{...}(k)$$

Thus bending moment and bending stress is zero at four points, one in each quadrant. Substituting the value of M_1 in Eqn. (*g*) from (*h*), we get

$$\varepsilon_0 = \frac{W}{AE}\left[\frac{R^2}{\pi(R^2+h^2)}\right] \qquad \text{...}(l)$$

$$\therefore \quad \varepsilon = \varepsilon_0+(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\left(\frac{y}{1+y/R}\right)$$

and $\sigma = E\varepsilon$

From Eqn. (*b*), we have

$$(1+\varepsilon_0)\left(\frac{1}{R_1}-\frac{1}{R}\right) = \frac{M_1}{EAh^2}+\frac{WR}{2Ah^2E}(1-\sin\theta)$$

$$\therefore \quad \varepsilon = \varepsilon_0+\left\{\frac{M_1}{EAh^2}+\frac{WR}{2Ah^2E}(1-\sin\theta)\right\}\left(\frac{Ry}{R+y}\right)$$

$$= \frac{W}{AE}\left\{\frac{R^2}{\pi(R^2+h^2)}\right\}+\left\{\frac{WR}{2}\left[\frac{2}{\pi}\left(\frac{R^2}{R^2+h^2}\right)-1\right]\times\frac{1}{EAh^2}+\frac{WR}{2Ah^2E}(1-\sin\theta)\right\}\left(\frac{Ry}{R+y}\right)$$

$$\therefore \quad \sigma = E\varepsilon = \frac{W}{A}\left(\frac{R^2}{\pi(R^2+h^2)}\right)+\left\{\frac{WR}{2Ah^2}\left[\frac{2R^2}{\pi(R^2+h^2)}-1\right]+\frac{WR}{2Ah^2}(1-\sin\theta)\right\}\left(\frac{Ry}{R+y}\right)$$

$$= \frac{W}{A}\left[\frac{R^2}{\pi(R^2+h^2)}+\frac{R^2}{2h^2}\left\{\frac{2R^2}{\pi(R^2+h^2)}-\sin\theta\right\}\times\left(\frac{Ry}{R+y}\right)\right]$$

$$= \frac{W}{A}\left[\frac{R^2}{\pi(R^2+h^2)}+\frac{R^2}{2h^2}\left\{\frac{2R^2}{\pi(R^2+h^2)}-\sin\theta\right\}\times\left(\frac{Ry}{R+y}\right)\right] \qquad \text{...}(m)$$

$$\text{Direct stress, } \sigma_d = \frac{W\sin\theta}{2A} \qquad \text{...}(n)$$

Resultant stress $\sigma_d = \sigma_d \pm \sigma$

(*i*) On a section taken along the line of action of W, $\theta = 0°$ and the stresses become :

(*a*) At outside of ring

$$\sigma_r = \frac{W}{\pi A}\left(\frac{R^2}{R^2+h^2}\right)\left[1+\frac{R^2}{h^2}\left(\frac{y_2}{R+y_2}\right)\right]$$

and is tensile in nature.

(*b*) At inside of ring

$$\sigma_r = \frac{W}{\pi A}\left(\frac{R^2}{R^2+h^2}\right)\left[\frac{R^2}{h^2}\left(\frac{y_1}{R-y_1}\right)-1\right]$$

(*ii*) On a section perpendicular to the line of action of W, $\theta = 90°$, and the stresses become

(*a*) At outside of ring

$$\sigma_r = -\frac{W}{A}\left[\frac{R^2}{\pi(R^2+h^2)} + \frac{R^2}{2h^2}\left\{\frac{2R^2}{\pi(R^2+h^2)} - 1\right\} \times \left(\frac{y_2}{R+y_2}\right)\right] + \frac{W}{2A}$$

and is compressive in nature.

(*b*) At inside of ring

$$\sigma_r = \frac{W}{A}\left[\frac{R^2}{2h^2}\left\{\frac{2R^2}{\pi(R^2+h^2)} - 1\right\}\left(-\frac{y_1}{R-y_1}\right) + \frac{R^2}{\pi(R^2+h^2)}\right] + \frac{W}{2A}$$

and is tensile in nature.

9.4 STRESSES IN A CHAIN LINK

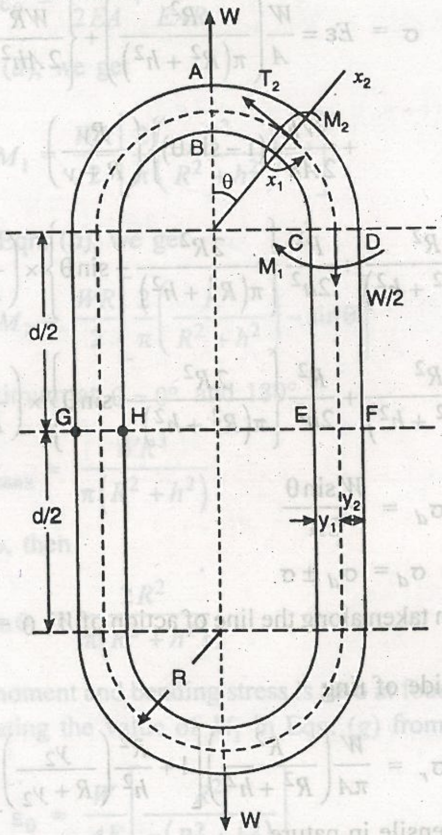


Fig 9.3

Consider a chain link as shown in Fig. 9.3. Let R be the mean radius of the semi circular ends and a the length of the straight sides. Consider the equilibrium of the portion x_1CDx_2 of the link.

$$M_2 = M_1 + \frac{WR}{2}(1-\sin\theta) \quad \text{...(a)}$$

Also

$$M_2 = E(1+\varepsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)Ah^2 \quad \text{...(b)}$$

Hence by comparing Eqns. (*a*) and (*b*), we get

$$E(1+\varepsilon_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)Ah^2 = M_1 + \frac{WR}{2}(1-\sin\theta) \quad \text{...(c)}$$

Multiplying both sides by $Rd\theta$ and integrate from 0 to $\pi/2$

$$E\int_0^{\pi/2}(1+\varepsilon_0)Ah^2\frac{R}{R_1}d\theta - E\int_0^{\pi/2}(1+\varepsilon_0)\times Ah^2 d\theta$$

$$= \int_0^{\pi/2} M_1 R d\theta + \int_0^{\pi/2}\frac{WR^2}{2}(1-\sin\theta)d\theta$$

Slope of the tangent at $L = \dfrac{M_1 a/2}{EI}$

$$\therefore \quad \int_0^{\pi/2}\frac{R(1+\varepsilon_0)}{R_1}d\theta = \frac{\pi}{2} - \frac{M_1 a}{2EI}$$

$$\therefore \quad EAh^2\left(\frac{\pi}{2} - \frac{M_1 a}{2EI}\right) - E(1+\varepsilon_0)Ah^2\frac{\pi}{2}$$

$$= \frac{\pi}{2}M_1 R + \frac{WR^2}{2}\left(\frac{\pi}{2} - 1\right)$$

$$M_1\left(\frac{\pi}{2}R + \frac{Aah^2}{2I}\right) = \frac{WR^2}{2}\left(1 - \frac{\pi}{2}\right) - \frac{\pi}{2}EAh^2\varepsilon_0 \quad \text{...(d)}$$

Now

$$\varepsilon_0 = \frac{1}{EA}\left(\frac{W}{2} + \frac{M_1}{R}\right) \quad \text{...(31)}$$

Substituting Eqn. (31) in (*d*), we get

$$M_1\left(\frac{\pi}{2}R + \frac{Aah^2}{2I} + \frac{\pi}{2}\frac{h^2}{2}\right) = \frac{WR^2}{2}\left(1 - \frac{\pi}{2}\right) - \frac{\pi}{4}Wh^2$$

Now $I = Ak^2$

where k = radius of gyration.

$$\therefore \quad M_1 = \frac{\frac{WR^2}{2}\left(1-\frac{\pi}{2}\right)-\frac{\pi}{4}Wh^2}{\frac{\pi}{2}R+\frac{ah^2}{2k^2}+\frac{\pi}{2}\frac{h^2}{R}}$$

$$= \frac{W\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}\right)}{R+\frac{ah^2}{\pi k^2}+\frac{h^2}{R}}$$

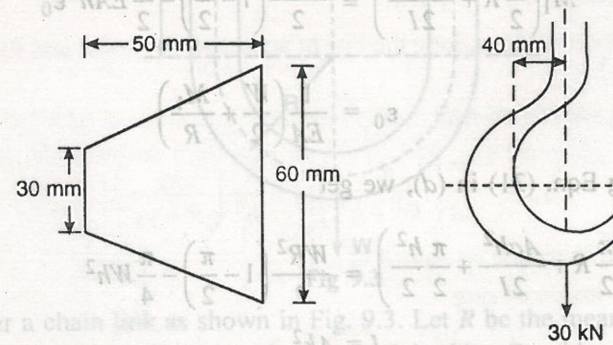
$$M_2 = \frac{W\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}\right)}{R+\frac{ah^2}{\pi k^2}+\frac{h^2}{R}}+\frac{WR}{2}(1-\sin\theta) \qquad \text{...(33)}$$

Substituting Eqn. (32) in 31), we get

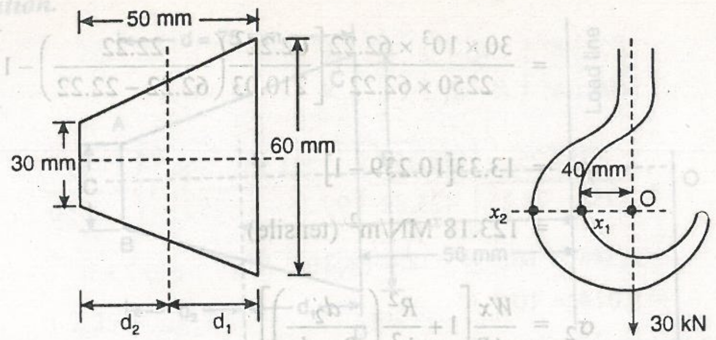
$$\varepsilon_0 = \frac{1}{EA}\left[\frac{W}{2}+\frac{\frac{W}{2}\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{h^2}{2}\right)}{R+\frac{ah^2}{\pi k^2}+\frac{h^2}{R}}\right] \qquad \text{...(34)}$$

$$\therefore \quad \sigma = E\varepsilon_0+\frac{WR}{2Ah^2}\left[\frac{2}{\pi}\frac{R^2}{\left(R^2+h^2\right)}-\sin\theta\right]\left(\frac{Ry}{R+y}\right)+\frac{W\sin\theta}{2A} \qquad \text{...(35)}$$

Example 9.1 A central horizontal section of hook is a symmetrical trapezium 50 mm deep, the inner width being 60 mm and the outer width being 30 mm. Calculate the extreme intensities of stress, when the hook carries a load of 30 kN, the load line passing 40 mm from the inside edge of the section and the centre of curvature being in the load line. (UPTU 2002-03)



Solution. Here B = 60 mm, C = 30 mm, d = 50 mm, W = 30 kN, $R_1 = Ox_1$ = 40 mm



Area of the cross section

$$A = \left(\frac{60+30}{2}\right)\times 50 = 2250 \text{ mm}^2$$

$$d_1 = \frac{d}{3}\left(\frac{B+30}{B+C}\right) = \frac{50}{3}\left(\frac{60+60}{60+30}\right) = 22.22 \text{ mm}$$

$$d_2 = d - d_1 = 50 - 20.22 = 27.78 \text{ mm}$$

$$\therefore \quad R = R_1 + d_1 = 40 + 22.2 = 62.22 \text{ mm}$$

Here $\quad x = R = 62.22$ mm

$$h^2 = \frac{R^3}{A}\left[C\ln\left(\frac{F+d_2}{R-d_1}\right)+\left(\frac{B-C}{d}\right)\times(R+d_2)\right.$$

$$\left.\ln\left(\frac{R+d_2}{R-d_1}\right)-(B-C)-R^2\right]$$

$$= \frac{62.22^3}{2250}\left[30\ln\left(\frac{62.22+27.78}{62.22-22.22}\right)+\left(\frac{60-30}{50}\right)(62.22+27.78)\right.$$

$$\left.\ln\left(\frac{62.22+27.78}{62.22-22.22}\right)-(60-30)\right]-(62.22)^2$$

$$= 107.055[30\ln 2.25+54\ln 2.25-30]-3871.33$$

$$= 107.055\,[30\times 0.811+54\times 0.811-30]-3871.33$$

$$= 210.03$$

Bending stresses are

$$\sigma_1 = \frac{Wx}{AR}\left[\frac{R^2}{h^2}\left(\frac{d_1}{R-d_1}\right)-1\right]$$

$$= \frac{30 \times 10^3 \times 62.22}{2250 \times 62.22}\left[\frac{62.22^2}{210.03}\left(\frac{22.22}{62.22 - 22.22}\right) - 1\right]$$

$$= 13.33[10.239 - 1]$$

$$= 123.18 \text{ MN/m}^2 \text{ (tensile)}$$

$$\sigma_2 = \frac{Wx}{AR}\left[1 + \frac{R^2}{h^2}\left(\frac{d_2}{R + d_2}\right)\right]$$

$$= \frac{30 \times 10^3 \times 62.22}{2250 \times 62.22}\left[1 + \frac{62.22^2}{210.03}\left(\frac{27.78}{62.22 - 27.78}\right)\right]$$

$$= 89.186 \text{ MN/m}^2 \text{ (compressive)}$$

Direct stress $\sigma_d = \frac{W}{A} = \frac{30 \times 10^3}{2250} = 13.33 \text{ MN/m}^2$

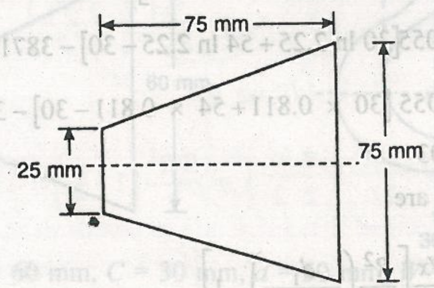
$\therefore$ Resultant stress on the inside fibres

$$= \sigma_1 + \sigma_d = 123.18 + 13.33 = 136.51 \text{ MN/m}^2 \textbf{ Ans.}$$

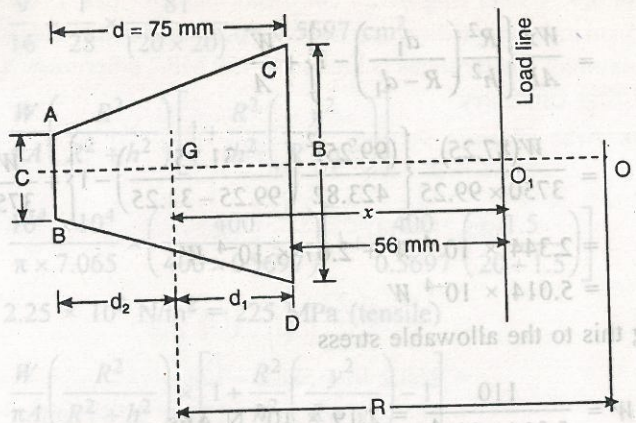
Resultant stress on the outside fibres

$$= \sigma_2 - \sigma_d = 89.186 - 13.33 = 75.86 \text{ MN/m}^2 \textbf{ Ans.}$$

Example 9.2 The principal section of a hook is a symmetrical trapezium as shown in figure. The centre of curvature of the centroidal axis, at the principal section, is in the plane of the section and is 68 mm from the inside of it. The load line passes 56 mm from the inner side of the section. If the maximum allowable stress is 110 MN/m^2, estimate the safe load for this hook.



Solution.



Here $B = 75$ mm, $C = 25$ mm, $d = 75$ mm, $R = 56 + d_1$

Area of cross section A

$$= \left(\frac{B + C}{2}\right)d = \left(\frac{75 + 25}{2}\right)75 = 3750 \text{ mm}^2$$

$$d_1 = \frac{d}{3}\left(\frac{B + 2C}{B + C}\right) = \frac{75}{3}\left(\frac{75 + 50}{75 + 25}\right) = 31.25 \text{ mm}$$

$$d_2 = 75 - 31.72 = 43.75 \text{ mm}$$

$$R = 68 + d_1 = 68 + 31.25 = 99.25 \text{ mm}$$

$$x = 56 + d_1 = 56 + 31.25 = 87.25 \text{ mm}$$

$$h^2 = \frac{(99.25)^3}{3750}\left[25 \ln\left(\frac{99.25 + 43.75}{99.25 - 31.25}\right) + \left(\frac{75 - 25}{75}\right)(99.25 + 43.75)\right.$$

$$\left.\ln\left(\frac{99.25 + 43.75}{99.25 - 31.25}\right) - (75 - 25)\right] - (99.25)^2$$

$$= 260.71\,[25 \ln 2.1029 + 95.33 \ln 2.1029 - 50] - 9850.56$$

$$= 260.71\,[25 \times 0.7433 + 95.33 \times 0.7433 - 50] - 9850.56$$

$$= 10283.38 - 9850.56 = 432.82$$

Let the allowable load on the hook be W Newton

then $M = Wx$

$$= 87.25\ W \text{ N-mm}$$

The maximum tensile stress will occur at points along DC

∴ Resultant stress along DC

$$= \frac{Wx}{AR}\left\{\frac{R^2}{h^2}\left(\frac{d_1}{R-d_1}\right)-1\right\}+\frac{W}{A}$$

$$= \frac{W(87.25)}{3750\times 99.25}\left\{\frac{(99.25)^2}{423.82}\left(\frac{31.25}{99.25-31.25}\right)-1\right\}+\frac{W}{3750}$$

$$= 2.344 \times 10^{-4}\, W + 2.67 \times 10^{-4}\, W$$

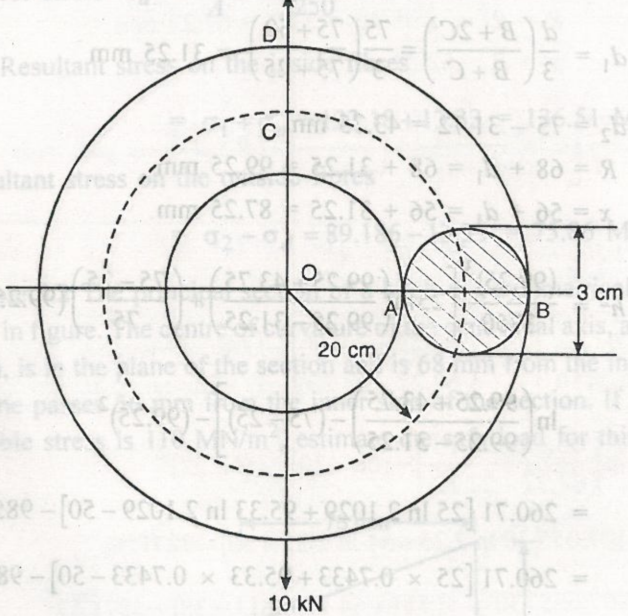
$$= 5.014 \times 10^{-4}\, W$$

Equating this to the allowable stress

$$W = \frac{110}{5.014\times 10^{-4}} = 2.19 \times 10^5 \text{ N } \textbf{Ans.}$$

Example 9.3 A ring made of 3.0 cm diameter steel bar carries a pull of 10 kN. Calculate the maximum tensile and compressive stresses in the material of the ring. The mean radius of the ring is 20 cm.

Solution.



Area of cross section $A = \frac{\pi}{4}(3)^2 = 7.065$

$$h^2 = \frac{D^2}{16} + \frac{1}{128}\frac{D^4}{R^2}$$

$$= \frac{9}{16} + \frac{1}{28} \times \frac{81}{(20\times 20)} = 0.5697 \text{ cm}^2$$

$$\sigma_D = \frac{W}{\pi A}\left(\frac{R^2}{R^2+h^2}\right)\left[1+\frac{R^2}{h^2}\left(\frac{y^2}{R+y^2}\right)\right]$$

$$= \frac{10^4\times 10^4}{\pi\times 7.065}\times\left(\frac{400}{400\times 0.5697}\right)\left[1+\frac{400}{0.5697}\left(\frac{1.5}{20+1.5}\right)\right]$$

$$= 2.25 \times 10^8 \text{ N/m}^2 = 225 \text{ MPa (tensile)}$$

$$\sigma_C = \frac{W}{\pi A}\left(\frac{R^2}{R^2+h^2}\right)\times\left[1+\frac{R^2}{h^2}\left(\frac{y^2}{R-y_1}\right)-1\right]$$

$$= \frac{10^4\times 10^4}{\pi\times 7.065}\times\left(\frac{400}{400\times 0.5697}\right)\left[\frac{400}{0.5697}\left(\frac{1.5}{20+1.5}\right)-1\right]$$

$$= 2.5175 \times 108 \text{ N/m}^2 = 251.75 \text{ MPa (compressive)}$$

$$\sigma_A = -\frac{W}{A}\left[\frac{R^2}{2h^2}\left\{\frac{2R^2}{\pi(R^2+h^2)}-1\right\}\left(\frac{y^2}{R-y_1}\right)-\frac{R^2}{\pi(R^2+h^2)}\right]+\frac{W}{2A}$$

$$\frac{10^8}{7.065}\left[\frac{400}{2\times 0.5697}\left\{\frac{2\times 400}{\pi(400.5697)}-1\right\}\left(\frac{1.5}{20-1.5}\right)-\frac{400}{\pi(400.5697)}\right]+\frac{10^8}{2\times 7.065}$$

$$= 1.58077 \times 10^8 \text{ N/m}^2 = 158.08 \text{ MPa (tensile)}$$

$$\sigma_B = -\frac{W}{A}\left[\frac{R^2}{\pi(R^2+h^2)}+\frac{R^2}{2h^2}\left(\frac{2R^2}{\pi(R^2+h^2)}-1\right)\left(\frac{y^2}{R+y2}\right)\right]+\frac{W}{2A}$$

$$= -\frac{10^8}{7.065}\left[\frac{400}{\pi(400.5697)}+\frac{400}{2\times 0.5697}\left(\frac{2\times 400}{\pi\times 400.5697}\right)\left(\frac{1.5}{20+1.5}\right)\right]+\frac{10^8}{2\times 7.065}$$

$$= -2.179 \times 10^8 \text{ N/m}^2 = -217.9 \text{ MPa (compressive)}$$

∴ Maximum tensile stress = 225 MPa

Maximum compressive stress = 251.75 MPa

Example 9.4 A steel ring of 22 cm mean diameter has a rectangular cross section 5 cm in the radial direction and 3 cm perpendicular to the radial direction. If the maximum tensile stress is limited to 150 MPa, determine the tensile load that the ring can carry.

Solution. Area of cross section

$$A = 5 \times 3 = 15 \text{ cm}^2$$

$$h^2 = \frac{R^3}{D}\ln\left(\frac{2R+D}{2R-D}\right) - R^2$$

$$= \frac{11^3}{5}\ln\left(\frac{22+5}{22-5}\right) - (11)^2$$

$$= 266.2 \ln 1.5882 - 121$$

$$= 266.2 \times 0.4626 - 121 = 2.144 \text{ cm}^2$$

The maximum tensile stress occurs at $\theta = 0°$ on the outside of the ring

$$\sigma = \frac{W}{\pi A}\left(\frac{R^2}{R^2+h^2}\right)\left[1 + \frac{R^2}{h^2}\left(\frac{y_2}{R+y_2}\right)\right]$$

$$150 \times 10^6 = \frac{W \times 10^4}{\pi \times 15}\left(\frac{121}{121+2.144}\right)\left[1 + \frac{121}{2.144}\left(\frac{2.5}{11+2.5}\right)\right]$$

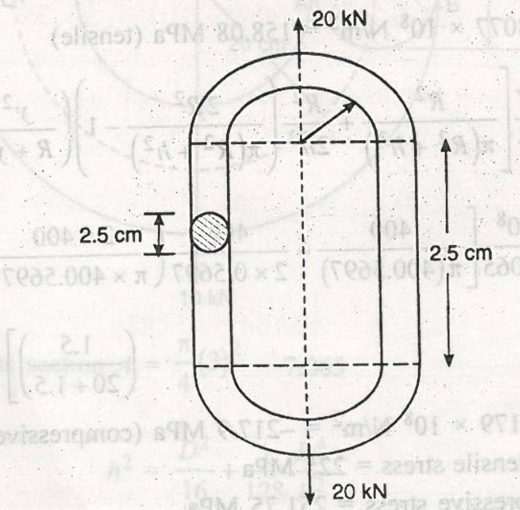
$$150 \times 10^6 = 2.3889 \times 10^3\, W$$

or $\quad W = 62.79 \times 10^3$

or $\quad W = 62.79$ kN

Example 9.5 A chain link is subjected to a pull of 20 kN. It is composed of steel 2.5 cm diameter and has a mean radius of 3 cm. Its semicircular ends are connected by straight pieces 2.5 cm long. Estimate maximum compressive stress in the link and tensile stress at the same section.

Solution.



$$A = \frac{\pi}{4} \times (2.5)^2 = 4.91 \text{ cm}^2$$

$$h^2 = \frac{D^2}{16} + \frac{D^4}{128R^2}$$

$$= \frac{(2.5)^2}{16} + \frac{(2.5)^4}{128 \times 9} = 0.4245 \text{ cm2}$$

$$M_1 = \frac{W\left(\frac{R^2}{\pi} - \frac{R^2}{2} - \frac{h^2}{2}\right)}{R + \frac{h^2 a}{K^2 \pi} + \frac{h^2}{R}}$$

Now $\quad I = 4.91K^2 = \frac{\pi \times (2.5)^4}{64}$

$\therefore \quad K^2 = 0.3905 \text{ cm}^2$

$$M_1 = \frac{20 \times 10^3\left(\frac{9}{\pi} - \frac{9}{2} - \frac{0.4245}{2}\right) \times 10^{-2}}{3 + \left(\frac{0.4245 \times 2.5}{\pi \times 0.3905}\right) + \left(\frac{0.4245}{3}\right)}$$

$$= -92.22 \text{ N-m}$$

Now $\quad \varepsilon_0 = \frac{1}{EA}\left(\frac{W}{2} + \frac{M_1}{R}\right)$

$$= \frac{10^4}{E4.91}\left(\frac{20 \times 10^3}{2} - \frac{92.218}{3 \times 10^{-2}}\right)$$

$$= \frac{14.106 \times 10^6}{E}$$

$$\sigma = E\varepsilon_0 + \frac{WR}{2Ah^2}\left[\frac{2}{\pi} - \frac{R^2}{\pi(R^2+h^2)}\sin\theta\right] \times \left(\frac{Ry}{R+y}\right) + \frac{W\sin\theta}{2A}$$

Compressive stress is maximum at $\theta = 0°$ on the inside part of the link

$\therefore \quad y = -1.25$ cm, $\theta = 0°$

$$\sigma = 14.106 \times 10^6 + \frac{20 \times 10^3 \times 3 \times 10^8}{2 \times 4.91 \times 0.4245}\left[\frac{2}{\pi} \times \frac{9}{9.4245}\right]\left(\frac{3 \times 1.25}{1.75}\right)$$

$$= 14.106 + (-187.508) = -173.402 \text{ MPa } \textbf{Ans.}$$

Tensile stress at this location (on the outside surface)

$$\sigma = 14.106 + \frac{WR}{2Ah^2}\left[\frac{2}{\pi}\frac{R^2}{\left(R^2+h^2\right)}\right]\left(\frac{Ry_2}{R+y_2}\right)$$

$$= 14.106 + \frac{20\times 10^3 \times 3\times 10^8}{2\times 4.91\times 0.4245}\left(\frac{2}{3.14}\times\frac{9}{9.4245}\right)\left(\frac{3\times 1.25}{4.25}\right)$$

$= 91.31$ MPa **Ans.**

EXPECTED DERIVATIONS

(*i*) A chain link made of circular section has the dimensions shown. Prove that if d, the diameter of the section, is assumed small compared with R, then the maximum bending moment occurs at the point of application of the load and is equal to

$$\frac{PR}{2}\frac{(l+2R)}{l+\pi R}$$

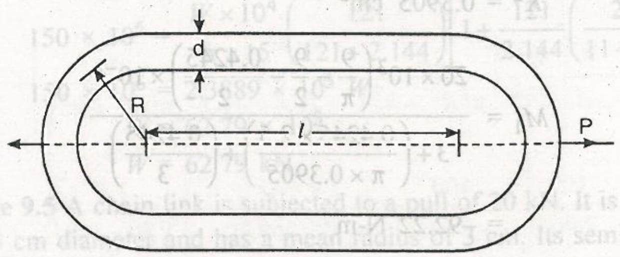


Fig. 9.4

(*ii*) For the crane hook shown in Fig. 9.1 O is the centre of curvature and the load line passes through O_1. The radius of curvature is R. Deduce the expressions for bending stresses at point x_1 and x_2.

(*iii*) Consider a circular ring loaded as shown in Fig. 9.2. Find the resultant stresses at outside and inside of ring on a section taken along the line of action of W.

(*iv*) Consider a chain link as shown in Fig. 9.3. R is the mean radius of semi circular ends and a is the length of the straight sides. Derive the expressions for the stresses on the inside and outside surfaces.

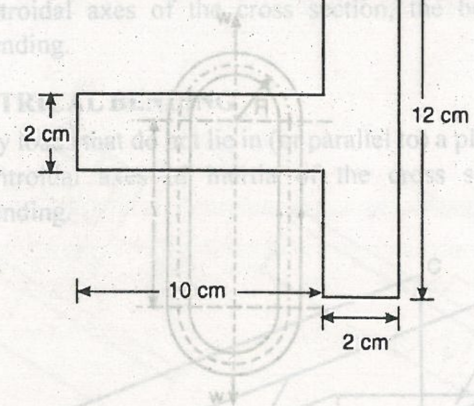
NUMERICAL PROBLEMS

1. A crane hook is of trapezoidal cross section having inner side 80 mm, outer side 300 mm and depth 120 mm. The radius of curvature of the inner side is 80 mm. If a load of 100 kN is applied to the hook passing through the centre of curvature, determine the maximum tensile and compressive stresses at the critical cross section. **[Ans.** 141.9 MPa, – 74.8 MPa]

2. Determine the load carrying capacity of a hook of rectangular cross section. The thickness of the hook is 75 mm, the radius of the inner fibres is 150 mm, while that of the outer fibres is 250 mm. The line of action of the forces passes at a distance of 75 mm from the inner fibres. The allowable stress is 70 MPa. **[Ans.** 52.51 kN]

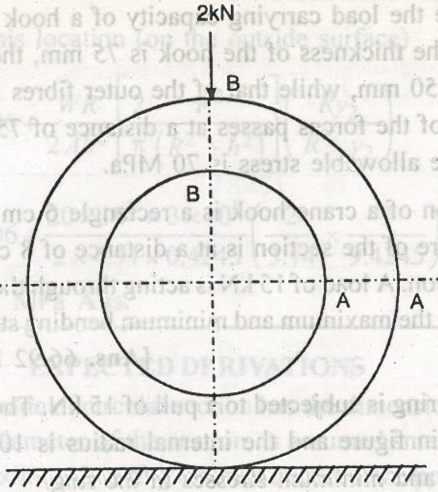
3. The section of a crane hook is a rectangle 6 cm × 4 cm. The centre of curvature of the section is at a distance of 8 cm from the centroid of the section. A load of 15 kN is acting through the centre of curvature. Determine the maximum and minimum bending stresses induced in the hook. **[Ans.** 66.92 MPa, – 39.51 MPa]

4. A circular ring is subjected to a pull of 15 kN. The ring is of *T*-section as shown in figure and the internal radius is 10 cm. Determine the maximum and minimum stresses in the ring. **[Ans.** 7.36 MPa, – 5.26 MPa]

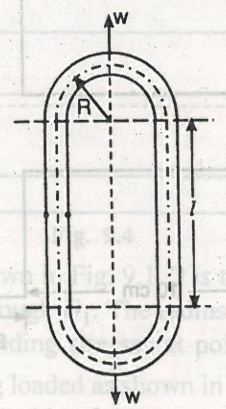


5. A chain link is subjected to a pull of 15 kN. It is composed of steel 2 cm diameter and has a mean radius of 2.5 cm. Its semi circular ends are connected by straight pieces 2.5 cm long. Estimate the maximum compressive stress in the link and the tensile stress at the same section. **[Ans.** – 207.95 MPa, 113.23 MPa]

6. A ring with a mean radius of curvature of 25 mm is subjected to a load of 200 N as shown in figure. The ring is made of circular section of 10 mm radius. Calculate the circumferential stress on the inside of the fibre of the ring at A and B. **[Ans.** σ_A = – 19.9 N/mm², σ_B = 29.1 N/mm²]



7. A chain link as shown in figure is made of round steel rod of 6 mm diameter. If R = 25 mm and l = 45 mm, calculate the ratio of the maximum tensile stress at the section where load is applied to that at the section half way along the straight portion. [Ans. 2.68]



10

Unsymmetrical Bending

10.1 INTRODUCTION

Frequently beams are of unsymmetric cross section, or even if the cross section is symmetric, the plane of the applied loads may not be the one of the planes of symmetry. In either of these cases the expression $\sigma = My/I$ is not valid for determination of the bending stress. There are situations when the plane of loading (plane of bending) does not lie in (or parallel to) a plane that contains the principal centroidal axes of the cross section, the bending is called unsymmetrical bending.

10.2 UNSYMMETRICAL BENDING

Bending caused by loads that do not lie in (or parallel to) a plane that contains the principal centroidal axes of inertia of the cross section is called unsymmetrical bending.

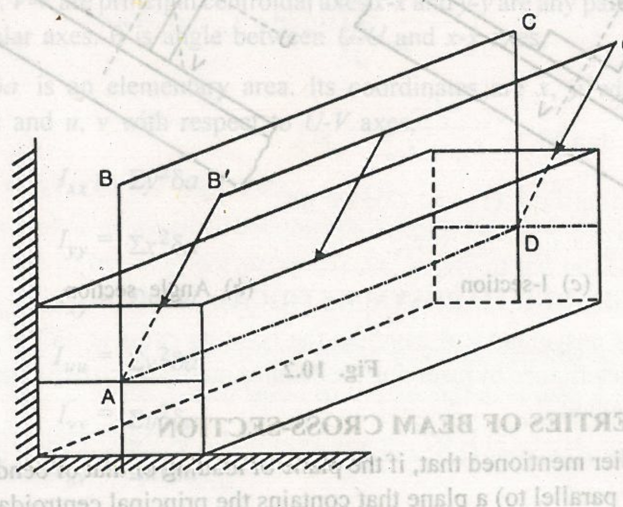
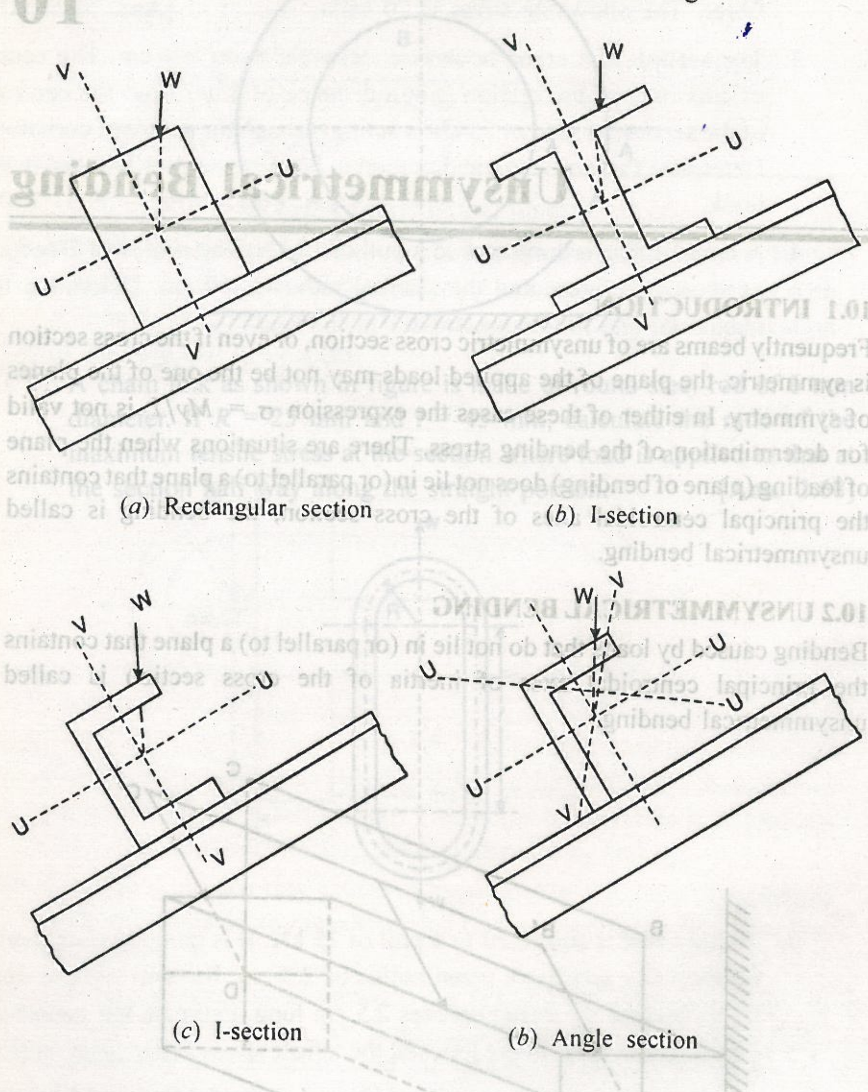


Fig. 10.1

In Fig. 10.1, ABCD is the plane containing the principal centroidal axes of inertia and plane $A'B'C'D$ is the plane containing the loads. These loads will cause unsymmetrical bending.

Some cases of unsymmetrical bending are shown in Fig. 10.2.



(*a*) Rectangular section (*b*) I-section

(*c*) I-section (*b*) Angle section

Fig. 10.2

10.3 PROPERTIES OF BEAM CROSS-SECTION

We have earlier mentioned that, if the plane of loading or that of bending does not lie in (or parallel to) a plane that contains the principal centroidal axes of beam cross section, the bending is called unsymmetrical bending. In this article we shall discuss about the centroidal principal axes of a beam cross-section.

The *centroidal principal axes* of a section are defined as a pair of rectangular axes through the centre of gravity of a plane area, such that the product of inertia is zero.

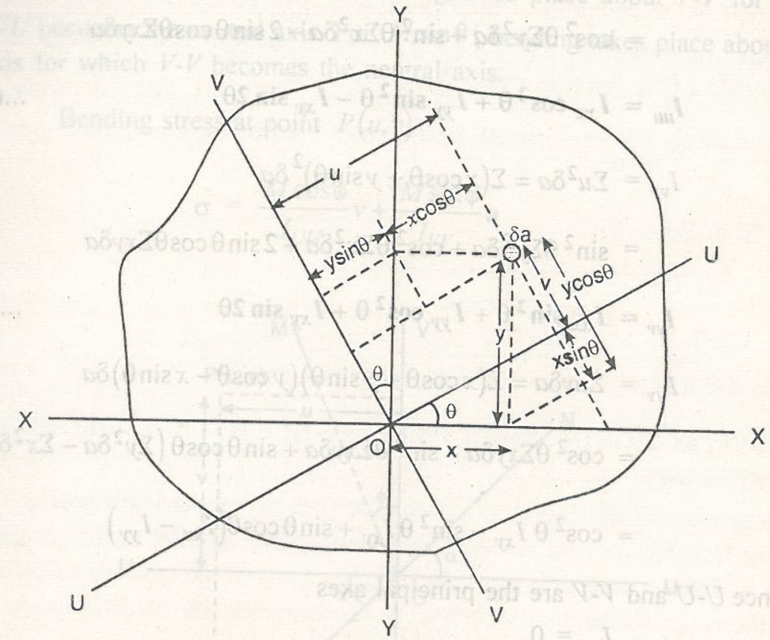


Fig. 10.3

Determination of Centroidal Principal Axes of a Section

Let *U-U*, *V-V* are principal centroidal axes *x-x* and *y-y* are any pair of centroidal rectangular axes. θ is angle between *U-U* and *x-x* axes.

Let δa is an elementary area. Its coordinates are *x*, *y* with respect to *x-y* axes and *u*, *v* with respect to *U-V* axes.

Now $I_{xx} = \Sigma y^2 \delta a$

$$I_{yy} = \Sigma x^2 \delta a$$

$$I_{xy} = \Sigma xy \delta a$$

$$I_{uu} = \Sigma v^2 \delta a$$

$$I_{vv} = \Sigma u^2 \delta a$$

$$I_{uv} = \Sigma uv \delta a$$

Now $u = x\cos\theta + y\sin\theta$

$$v = y\cos\theta + x\sin\theta$$

$$\therefore \quad I_{uu} = \Sigma v^2 \delta a$$

$$= \Sigma (y\cos\theta - x\sin\theta)^2 \delta a$$

$$= \cos^2\theta \Sigma y^2 \delta a + \sin^2\theta \Sigma x^2 \delta a - 2\sin\theta\cos\theta \Sigma xy\delta a$$

$$\Rightarrow \quad I_{uu} = I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - I_{xy}\sin 2\theta \quad \ldots(1)$$

$$I_{vv} = \Sigma u^2\delta a = \Sigma(x\cos\theta + y\sin\theta)^2 \delta a$$

$$= \sin^2\theta \Sigma y^2\delta a + \cos^2\theta \Sigma x^2 \delta a + 2\sin\theta\cos\theta\Sigma xy\delta a$$

$$I_{vv} = I_{xx}\sin^2\theta + I_{yy}\cos^2\theta + I_{xy}\sin 2\theta \quad \ldots(2)$$

$$I_{uv} = \Sigma uv\delta a = \Sigma(x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\delta a$$

$$= \cos^2\theta\Sigma xy\delta a - \sin^2\theta \Sigma xy\delta a + \sin\theta\cos\theta\left(\Sigma y^2\delta a - \Sigma x^2 \delta a\right)$$

$$= \cos^2\theta\, I_{xy} - \sin^2\theta\, I_{xy} + \sin\theta\cos\theta\left(I_{xx} - I_{yy}\right)$$

Since *U-U* and *V-V* are the principal axes

$$\therefore \quad I_{uv} = 0$$

$$\Rightarrow \quad \cos^2\theta\, I_{xy} - \sin^2\theta\, I_{xy} + \sin\theta\cos 2\theta\left(I_{xx} - I_{yy}\right) = 0$$

$$\Rightarrow \quad \left(\frac{I_{xx} - I_{yy}}{2}\right)\sin 2\theta + I_{xy}\cos 2\theta = 0$$

$$\Rightarrow \quad \tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} \quad \ldots(3)$$

Note : Adding Eqn. (1) and (2), we get

$$I_{xx} + I_{yy} = I_{uu} + I_{vv} \quad \ldots(4)$$

10.4 STRESSES DUE TO UNSYMMETRICAL BENDING

In the case of unsymmetrical bending, the bending stress at an point in the beam can be determined by resolving the bending moment into two components along principal axes.

Let the plane of bending (M) be inclined at an angle ϕ with one of the principal planes.

M can be resolved in component $M\cos\theta$ along plane *V-V* and $M\sin\theta$ along the plane *U-U*.

Once M is resolved in two components, the simple theory of bending can be applied to bending occurring in the principal planes.

For the component $M\cos\theta$, bending takes place about *V-V* for which *U-U* becomes the neutral axis. For $M\sin\theta$, bending takes place about *U-U* axis for which *V-V* becomes the neutral axis.

$\therefore$ Bending stress at point $P(u,v)$

$$\sigma = \frac{M\cos\phi}{I_{UU}}v + \frac{M\sin\phi}{I_{VV}}u \quad \ldots(5)$$

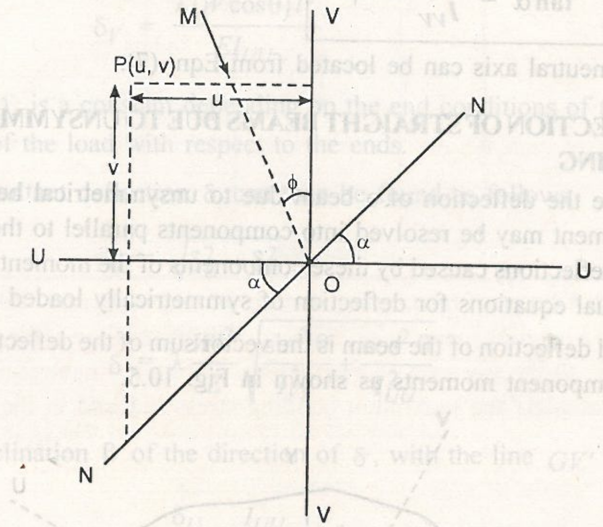


Fig. 10.4

10.5 LOCATION OF NEUTRAL AXIS

In the case of unsymmetrical bending, the neutral axis is neither perpendicular to the plane of bending, nor perpendicular to any of the principal planes.

In Fig. 10.4,

ϕ = inclination of the plane of bending to *V-V* axis

α = inclination of the neutral axis with the *U-U* axis.

On any point (such as P) on neutral axis, bending stress σ will be zero.

Equating Eqn. (5) to zero,

$$\sigma = 0 = \frac{M\cos\phi}{I_{UU}}v + \frac{M\sin\phi}{I_{VV}}u$$

$$\Rightarrow \qquad v = -u\frac{I_{UU}}{I_{VV}}\tan\phi \qquad \text{...(6)}$$

Eqn. (6) is the equation of the neutral axis *N-N* which is a straight line.

It is clear that when $v = 0$, then $u = 0$, hence the neutral axis passes through the centroid of the section. Now from Fig. 10.4

$$\tan\alpha = -\frac{v}{u}$$

But from Eqn. (6)

$$\boxed{\tan\alpha = \frac{I_{UU}}{I_{VV}}\tan\phi} \qquad \text{...(7)}$$

Thus the neutral axis can be located from Eqn. (7).

10.6 DEFLECTION OF STRAIGHT BEAMS DUE TO UNSYMMETRICAL BENDING

To determine the deflection of a beam due to unsymmetrical bending, the bending moment may be resolved into components parallel to the principal planes and deflections caused by these components of the moment calculated from the usual equations for deflection of symmetrically loaded beams.

The actual deflection of the beam is the vector sum of the deflections found from the component moments as shown in Fig. 10.5.

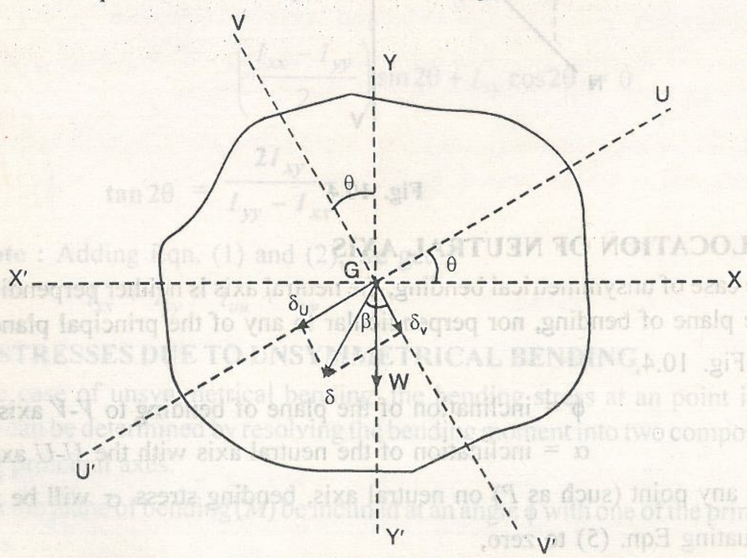


Fig. 10.5

It can be found that the direction of the deflection is always perpendicular to the neutral axis.

Fig. 10.5 shows a load W, acting along YY' line, on a section of a beam. Axes GU and GV are the principal axes of the section having G as the centroid. The load W can be resolved into two components, $(W\cos\theta)$ along GV and $(W\sin\theta)$ along GU'. The component $(W\cos\theta)$ will cause deflection δ_v along the line GV' due to bending about UU' axis and $(W\sin\theta)$ will deflect the beam by δ_U along the line GU' for its bending about VV' axis. Depending on the end conditions of the beam these deflection will be given by

$$\delta_U = \frac{\lambda(W\sin\theta)l^3}{EI_{VV}} \qquad \text{...(8)}$$

$$\delta_V = \frac{\lambda(W\cos\theta)l^3}{EI_{UU}} \qquad \text{...(9)}$$

where λ is a constant depending on the end conditions of the beam and position of the load with respect to the ends.

The resultant deflection δ can then be found as follows

$$\delta = \sqrt{\delta_U^2 + \delta_V^2}$$

$$\delta = \lambda\frac{Wl^3}{E}\sqrt{\frac{\sin^2\theta}{I_{VV}^2} + \frac{\cos^2\theta}{I_{UU}^2}} \qquad \text{...(10)}$$

The inclination β of the direction of δ, with the line GV' is given by

$$\tan\beta = \frac{\delta_U}{\delta_V} = \frac{I_{UU}}{I_{VV}}\tan\theta \qquad \text{...(11)}$$

10.7 SHORT NOTE ON NEUTRAL AXIS

There always exists one surface in the beam containing fibres that do not undergo any extension or compression, and thus are not subjected to any tensile or compressive stress. This surface is called the neutral surface of the beam. The intersection of the neutral surface with any cross section of the beam perpendicular to its longitudinal axis is called the *neutral axis*. All fibers on one side of the neutral axis are in a state of tension, while those on the opposite side are in compression.

When all fibers in the beam act within elastic range, the neutral axis passes through the centroid of the cross section.

Consider two sections mn and m_1n_1 at distance δx apart and subjected to bending moment M as shown in Fig. 10.6. The section mn and m_1n_1 which were vertical and parallel before the application of the moment will rotate through an angle θ after deformation and will remain straight. The fiber *ab* at the concave side of the beam shortens. Similarly the fiber *cd* on the concave side shortens and the fiber mn_1 on the convex side elongates. Also there exists one fiber such as *ef* the length of which remains unchanged and indicates that this undergoes neither extension nor compression. The layer *ef* is called the *neutral layer*. The line of intersection of neutral layer with the plane of cross section of the beam is called the *neutral axis*.

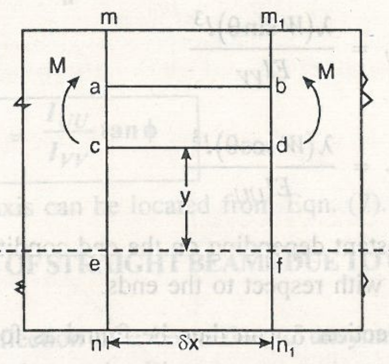


Fig. 10.6

Example 10.1 A beam of rectangular section, 80 mm wide and 120 mm deep is subjected to a bending moment of 12 kN-m. The trace of the plane of loading is inclined 45° to the y-y axis of the section. Locate the neutral axis of the section and calculate the maximum bending stress induced in the section. (UPTU 2002-03)

Solution.

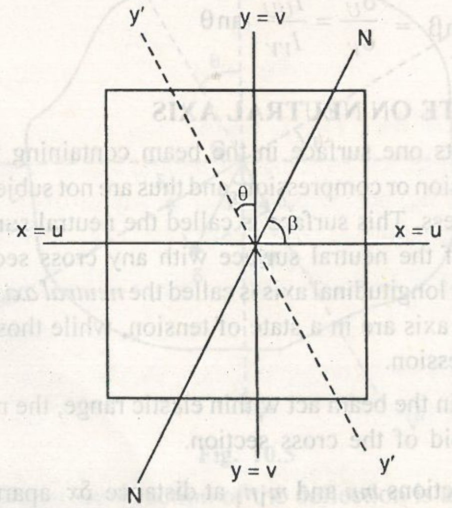


Fig. 10.7

Let the plane of loading be inclined at an angle θ with *y-y* axis and the neutral axis be inclined at β with the *x-x* axis.

$$\theta = 45^\circ$$

$$M = 12000 \times 1000 = 12 \times 10^6 \text{ Nmm}$$

$$I_x = I_u = \frac{1}{12} \times 80 \times 120^3 = 11.52 \times 10^6 \text{ mm}^4$$

$$I_y = I_v = \frac{1}{12} \times 120 \times 80^3 = 5.12 \times 10^6 \text{ mm}^4$$

$$\tan\beta = \frac{I_u}{I_v}\tan\theta = \frac{11.52 \times 10^6}{5.12 \times 10^6} \times \tan 45^\circ = 2.25$$

$$\Rightarrow \quad \beta = 66^\circ$$

This gives the location of the neutral axis.

Maximum stress will occur at point which is more distant from *NA*, either *B* or *D*.

$$\sigma = \frac{M\cos\theta}{I_u}v + \frac{M\sin\theta}{I_v}u$$

$$= \frac{M\cos\theta}{I_x}y + \frac{M\sin\theta}{I_y}x$$

where (x, y) are the coordinates of the point.

$$\sigma_B = -\frac{12 \times 10^6 \cos 45^\circ}{11.52 \times 10^6} \times 60 - \frac{12 \times 10^6 \sin 45^\circ}{5.12 \times 10^6} \times 40$$

$$= -110.5 \text{ N/mm}^2 \quad \text{(tensile)}$$

$$\sigma_D = +\frac{12 \times 10^6 \cos 45^\circ}{11.52 \times 10^6} \times 60 + \frac{12 \times 10^6 \sin 45^\circ}{5.12 \times 10^6} \times 40$$

$$= +110.5 \text{ N/mm}^2 \quad \text{(compressive)}$$

Example 10.2 A 5 cm by 3 cm by 0.5 cm angle is used as a cantilever of length 50 cm with 3 cm leg horizontal. A load of 1000 N is applied at the free end. Determine the position of the neutral axis.

Solution.

$$\bar{x} = \frac{(4.5 \times 0.5) \times 0.25 + (3 \times 0.5) \times 1.5}{(4.5 + 3) \times 0.5} = 0.75 \text{ cm}$$

$$\bar{y} = \frac{(4.5 \times 0.5) \times 2.75 + (3 \times 0.5) \times (0.25)}{(4.5 + 3) \times 0.5} = 1.75 \text{ cm}$$

$$I_x = \frac{0.5 \times 4.5^3}{12} + (0.5 \times 4.5) \times 1^2 + \frac{3 \times 0.5^3}{12} + (3 \times 0.5) \times 1.5^2$$

$$= 9.44 \text{ cm}^4$$

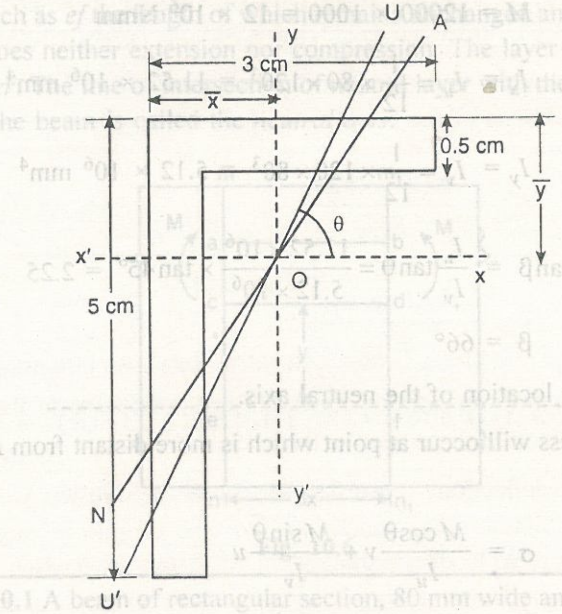


Fig. 10.8

$$I_y = \frac{4.5 \times 0.5^3}{12} + (4.5 \times 0.5) \times 0.5^2 + \frac{0.5 \times 3^3}{12} + (0.5 \times 3) \times 0.75^2$$

$$= 2.58 \text{ cm}^4$$

$$I_{xy} = (4.5 \times 0.5) \times (-0.5) \times (-1) + (3 \times 0.5)(0.75)(1.5)$$

$$= 2.813 \text{ cm}^4$$

$$\therefore \quad \tan 2\theta = \frac{2 \times 2.813}{2.58 - 9.44} = -0.820$$

$$\Rightarrow \quad \theta = 70°20'$$

$$I_u = \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y) \sec 2\theta$$

$$= \frac{1}{2}(9.44 + 2.58) + \frac{1}{2}(9.44 - 2.58) \sec 140°40'$$

$$= 1.59 \text{ cm}^4$$

$$I_v = I_x + I_y - I_u$$

$$= 9.44 + 2.58 - 1.59 = 10.43 \text{ cm}^4$$

$$M_v = 500000 \sin 70° 20' = 470080 \text{ N-mm}$$

$$M_u = 500000 \cos 70° 20' = 160830 \text{ N-mm}$$

$$\sigma = \frac{M_v u}{I_v} + \frac{M_u v}{I_u}$$

$$\Rightarrow \quad \sigma = \frac{470080}{104300} u + \frac{160830}{15900} v$$

$$= 4.51\, u + 10.6\, v \qquad (\because \sigma = 0 \text{ at neutral axis})$$

$$\therefore \quad 4.51u + 10.6v = 0$$

which is the equation for neutral axis.

This is a line through O inclined at $\tan^{-1}(-0.426)$

or $-23°4'$ to UU'

Example 10.3 A simply supported beam of *T*-section, 2.5 cm long carries a central concentrated load inclined at 30° to the *y*-axis as shown in Fig. 10.9. If the maximum compressive and tensile stresses in bending are not to exceed 75 MPa and 35 MPa respectively, find the maximum load the beam can carry.

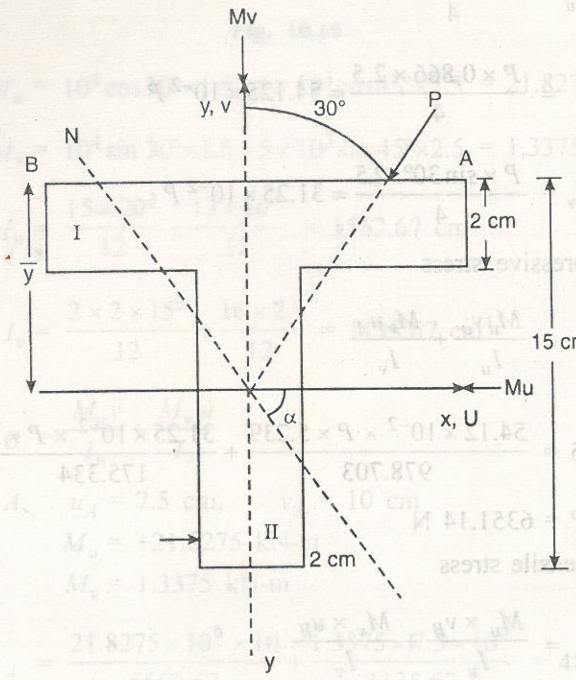


Fig. 10.9

Solution. Total area = 46 cm^2

$$\bar{y} = \frac{20 \times 1 + 26 \times 8.5}{46} = 5.239 \text{ cm}$$

$$I_x = \frac{10\times 2^3}{12} + 20\times(4.239)^2 + \frac{2\times 13^3}{12} + 26(8.5-5.239)^2$$

$$= 978.703 \text{ cm}^2$$

$$I_y = \frac{2\times 10^3}{12} + \frac{13\times 2^3}{12} = 175.334 \text{ cm}^2$$

$$I_{xy} = 0$$

$\therefore \quad I_x = I_u \quad$ and $\quad I_y = I_v$

Using $\tan\alpha = \dfrac{I_u}{I_v}\tan\phi$

$$\Rightarrow \tan\alpha = \frac{978.703}{175.334}\times\tan 30^\circ = 3.2228$$

$$\Rightarrow \alpha = 72.76^\circ$$

$$M_u = \frac{P\cos 30^\circ\times 2.5}{4}$$

$$= \frac{P\times 0.866\times 2.5}{4} = 54.125\times 10^{-2}P$$

$$M_v = \frac{P\times\sin 30^\circ\times 2.5}{4} = 31.25\times 10^{-2}P$$

Max. compressive stress

$$\sigma_A = \frac{M_u v_a}{I_u} + \frac{M_v u_A}{I_v}$$

$$\Rightarrow \quad 75 = \frac{54.12\times 10^{-2}\times P\times 5.239}{978.703} + \frac{31.25\times 10^{-2}\times P\times 5}{175.334}$$

$$\Rightarrow \quad P = 6351.14 \text{ N}$$

Maximum tensile stress

$$\sigma_B = \frac{M_u\times v_B}{I_u} + \frac{M_v\times u_B}{I_v}$$

$$\Rightarrow \quad 35 = (-0.28973 + 8.89116)\,P\times 10^{-2}$$

$$\Rightarrow \quad P = 5819.46 \text{ N}$$

$\therefore$ Permissible load = 5819.46 N

Example 10.4 A cantilever beam of I-section is used to support the loads inclined to the *V*-axis as shown in Fig. 10.10. Calculate the stresses at the corners *A*, *B*, *C* and *D*. Also locate the neutral axis.

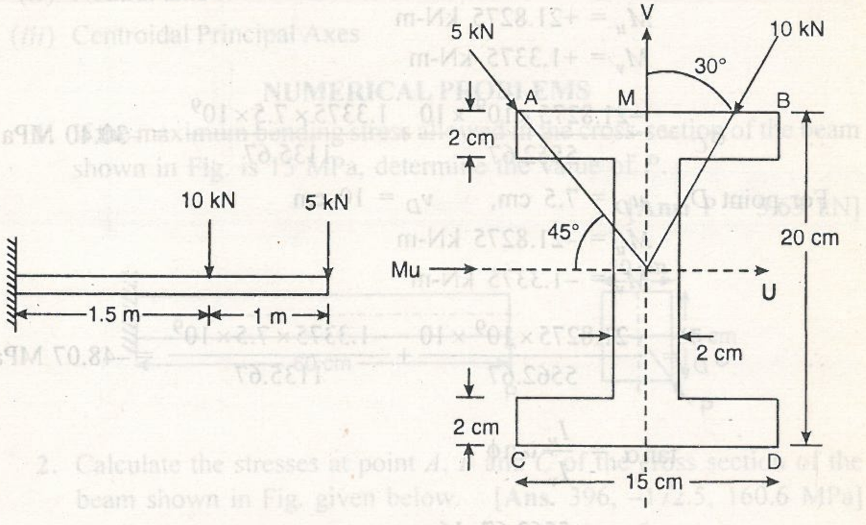


Fig. 10.10

Solution. $M_u = 10^4\cos 30^\circ\times 1.5 + 5\times 10^3\cos 45^\circ\times 2.5 = 21.8275$ kN-m

$$M_v = 10^4\sin 30^\circ\times 1.5 + 5\times 10^3\sin 45^\circ\times 2.5 = 1.3375 \text{ kN-m}$$

$$I_u = \frac{15\times 20^3}{12} - \frac{13\times 16^3}{12} = 5562.67 \text{ cm}^4$$

$$I_v = \frac{2\times 2\times 15^3}{12} + \frac{16\times 2^3}{12} = 1135.67 \text{ cm}^4$$

$$\sigma = \frac{M_u v}{I_u} + \frac{M_v u}{I_v}$$

For point *A*, $\quad u_A = 7.5$ cm, $\quad v_A = 10$ cm

$M_u = +21.8275$ kN-m

$M_v = 1.3375$ kN-m

$$\therefore \quad \sigma_A = \frac{21.8275\times 10^9\times 10}{5562.67} + \frac{1.3375\times 7.5\times 10^9}{1135.67} = 48.07 \text{ MPa}$$

For point *B*, $\quad u_B = 7.5$ cm, $\quad v_B = 10$ cm

$M_u = +21.8275$ kN-m

$M_v = -1.3375$ kN-m