$$\therefore \quad \sigma_B = \frac{21.8275 \times 10 \times 10^9}{5562.67} + \frac{1.3375 \times 7.5 \times 10^9}{1135.67} = 30.41 \text{ MPa}$$

For point C, $u_C = 7.5$ cm, $v_C = 10$ cm

$$M_u = +21.8275 \text{ kN-m}$$

$$M_v = +1.3375 \text{ kN-m}$$

$$\therefore \quad \sigma_C = \frac{-21.8275 \times 10^9 \times 10}{5562.67} + \frac{1.3375 \times 7.5 \times 10^9}{1135.67} = -30.40 \text{ MPa}$$

For point D, $u_D = 7.5$ cm, $v_D = 10$ cm

$$M_u = -21.8275 \text{ kN-m}$$

$$M_v = -1.3375 \text{ kN-m}$$

$$\therefore \quad \sigma_D = \frac{-21.8275 \times 10^9 \times 10}{5562.67} + \frac{-1.3375 \times 7.5 \times 10^9}{1135.67} = -48.07 \text{ MPa}$$

$$\tan\alpha = \frac{I_u}{I_v}\tan\phi$$

$$\Rightarrow \quad \tan\alpha = \frac{5562.67}{1135.67}\frac{M_u}{M_v}$$

$$= \frac{5562.67}{1135.67} \times \frac{1.3375}{21.8275} = 0.30014$$

$$\Rightarrow \quad \alpha = 16.70°$$

USEFUL RESULTS

1. $I_U = I_x \cos^2\theta + I_y \sin^2\theta - I_{xy}\sin 2\theta$
2. $I_V = I_x \sin^2\theta + I_y \cos^2\theta - I_{xy}\sin 2\theta$
3. $\tan 2\theta = \dfrac{2I_{xy}}{I_y - I_x}$
4. $I_x + I_y = I_u + I_v$
5. $\sigma = \dfrac{M\cos\phi}{I_u}v + \dfrac{M\sin\phi}{I_v}u = \dfrac{M_u v}{I_u} + \dfrac{M_v u}{I_v}$
6. $\tan\alpha = \dfrac{I_u}{I_v}\tan\phi$
7. $I_u = \dfrac{1}{2}(I_x + I_y) + \dfrac{1}{2}(I_x - I_y)\sec 2\theta$

REVIEW QUESTIONS

Write short notes on the following :

(*i*) Unsymmetrical Bending

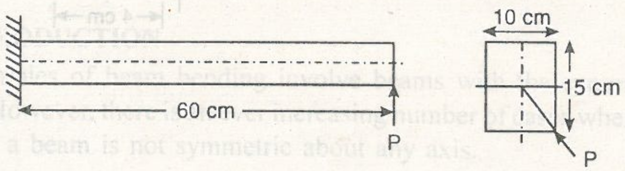
(*ii*) Neutral axis

(*iii*) Centroidal Principal Axes

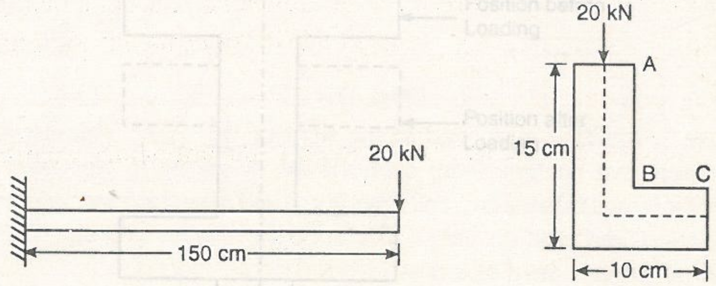
NUMERICAL PROBLEMS

1. If the maximum bending stress allowed in the cross-section of the beam shown in Fig. is 15 MPa, determine the value of P.

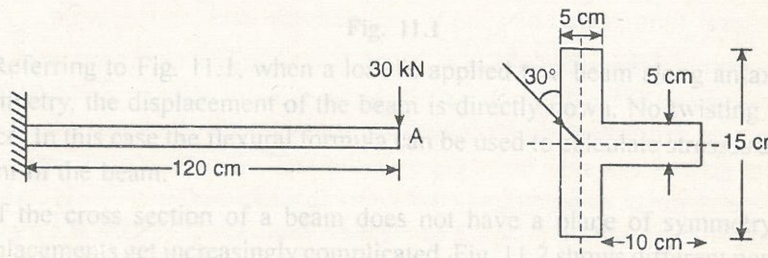
[**Ans.** P = 5.63 kN]



2. Calculate the stresses at point A, B and C of the cross section of the beam shown in Fig. given below. [**Ans.** 396, –172.5, 160.6 MPa]



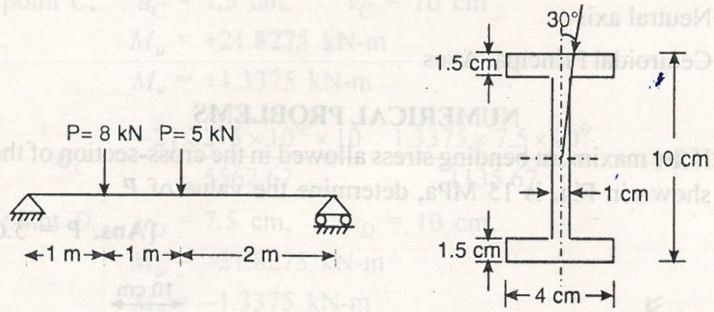
3. For the beam loaded as shown in Fig. determine the stresses at A and locate the neutral axis.



[**Ans.** 198.6 MPa, 21.1° clockwise from x-axis]

4. An *I*-beam section is loaded as shown in figure. Determine the stress at *A*. Also locate the position of the neutral axis.

[**Ans.** 385.4 MPa, 83°.23′ clockwise form x axis]



11 Shear Centre

11.1 INTRODUCTION

Most examples of beam bending involve beams with the symmetric cross sections. However, there is an ever increasing number of cases where the cross section of a beam is not symmetric about any axis.

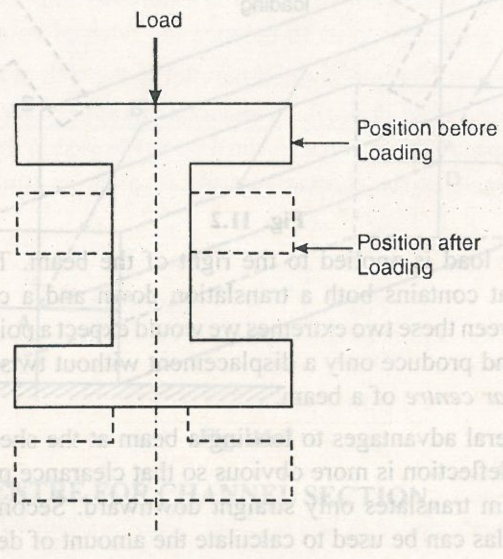


Fig. 11.1

Referring to Fig. 11.1, when a load is applied to a beam along an axis of symmetry, the displacement of the beam is directly down. No twisting takes place. In this case the flexural formula can be used to calculate stresses at any point in the beam.

If the cross section of a beam does not have a plane of symmetry, the displacements get increasingly complicated. Fig. 11.2 shows different possible loading situations for a non-symmetrical beam. In case 1 the load is applied to the left of the beam. In this case the displacement consists of both a translation down and also a counter-clockwise twist.

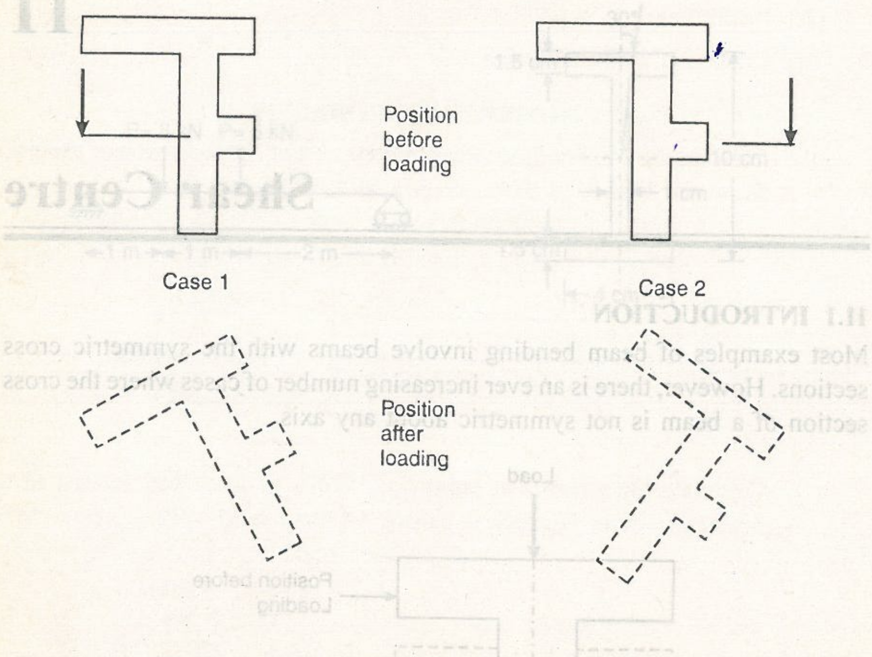


Fig. 11.2

In case 2, the load is applied to the right of the beam. This produces a displacement that contains both a translation down and a clockwise twist. Somewhere between these two extremes we would expect a point that we could apply the load and produce only a displacement without twisting. This point is called the *shear centre* of a beam.

There are several advantages to loading a beam at the shear centre. First, the path of any deflection is more obvious so that clearance problems can be avoided, the beam translates only straight downward. Second, the standard deflection formulas can be used to calculate the amount of deflection. Third, the flexural formula can be used to calculate the stress in the beam.

The simple flexure formula $\sigma = My/I$ is valid only if the transverse loads which give rise to bending act in a plane of symmetry of beam cross section. In this type of loading there is obviously no torsion of the beam. However, in more general cases the beam cross section will have no axes of symmetry and the problem of where to apply transverse loads so that the action is entirely bending with no torsion arises. Every elastic beam cross section has a point through which transverse forces may be applied so as to produce bending only with no torsion of the beam. The point is called the shear centre or *centre of flexure*.

The shear centre for any transverse section of the beam is the point of intersection of the bending axis and the plane of the transverse section. Shear centre is also called the *centre of twist*. If a beam has two axes of symmetry, then shear centre coincides with the centroid. If a load passes through the shear centre then there will be only bending in the cross-section and no twisting.

11.2 FLEXURAL AXIS OR BENDING AXIS

Flexural axis of a beam is the longitudinal axis through which the transverse bending loads must pass in order that the bending of the beam shall not be accompanied by twisting of the beam.

In Fig. 11.3, *ABCD* is a plane containing the principal centroidal axes of inertia and plane *AB′C′D* is the plane containing the loads. These loads will cause unsymmetrical bending.

In figure 11.3 *AD* is the flexural axis.

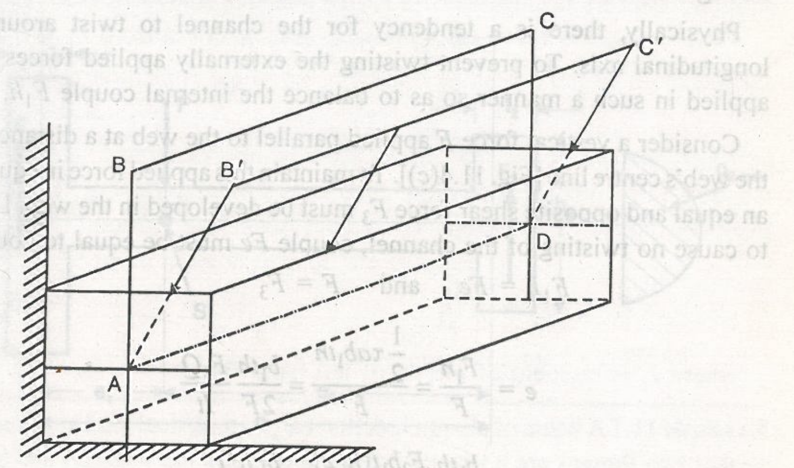


Fig. 11.3

11.3 SHEAR CENTRE FOR CHANNEL SECTION

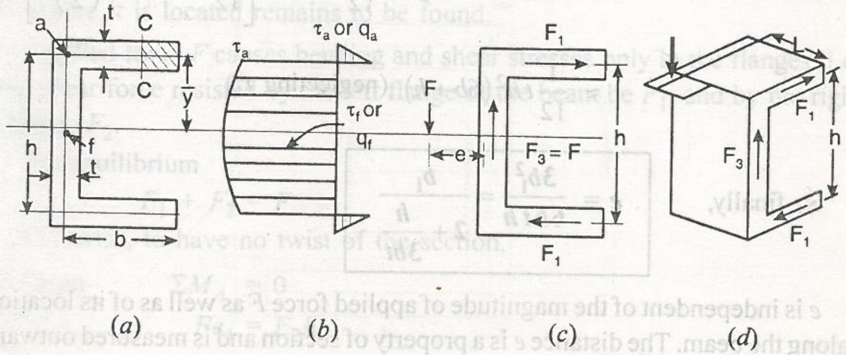


Fig. 11.4

Consider a beam having the cross section of a channel. Bending of this channel takes place around the horizontal axis. By taking an arbitrary cut at C-C in Fig. 11.4(a), q and τ may be found. The variation of q and τ is parabolic along the web. [Fig. 11.4(b)]

The average shear stress $\tau a/2$ multiplied by the areas of the flanges gives a force $F_1 = (\tau a/2)bt$, and the sum of the vertical shear stresses over the area of the web is

$$F_3 = \int_{-h/2}^{h/2} \tau t\, dy$$

These shear forces acting in the plane of cross section are shown in Fig. 11.4(c) and indicate that a force F_3 and a couple $F_1 h$ are developed at the section through the channel.

Physically, there is a tendency for the channel to twist around some longitudinal axis. To prevent twisting the externally applied forces must be applied in such a manner so as to balance the internal couple $F_1 h$.

Consider a vertical force F applied parallel to the web at a distance e from the web's centre line [Fig. 11.4(c)]. To maintain this applied force in equilibrium, an equal and opposite shear force F_3 must be developed in the web. Likewise, to cause no twisting of the channel, couple Fe must be equal to couple $F_1 h$.

$$F_1 h = Fe \quad \text{and} \quad F = F_3$$

$$e = \frac{F_1 h}{F} = \frac{\frac{1}{2}\tau a b_1 t h}{F} = \frac{b_1 t h}{2F}\frac{F_3 Q}{It}$$

$$= \frac{b_1 t h}{2F}\frac{F_3 b_1 t (h/2)}{It} = \frac{b_1^2 h^2 t}{4I}$$

$$I = I_{\text{web}} + 2I_{\text{flange}} = \frac{1}{12}th^3 + 2\left[\frac{1}{12}b_1 t^3 + b_1 t\left(\frac{h}{2}\right)^2\right]$$

$$\approx \frac{1}{12}th^2(6b+h) \text{ (neglecting } t^3)$$

So finally, $$\boxed{e = \frac{3b_1^2}{6bth} = \frac{b_1}{2+\dfrac{h}{3bt}}}$$

e is independent of the magnitude of applied force F as well as of its location along the beam. The distance e is a property of section and is measured outward from the centre of the web of the applied force.

The shear centre for any cross section lies on a longitudinal line parallel to the axis of the beam. Any transverse force applied through the shear centre causes no torsion of the beam. When a member of any cross sectional area is twisted, the twist takes place around the shear centre, which remains fixed. For this reason, the shear centre is sometimes called the centre of twist.

For the cross sectional areas having one axis of symmetry, the shear centre is always located on the axis of symmetry. For those that have two axes of symmetry the shear centre coincides with the centroid of the cross sectional area (case of I-beam).

The usual procedure of locating the shear centre consists of determining the shear forces, as F_1 and F_3, at a section and then finding the location of the external force necessary to keep these forces in equilibrium.

11.4 SHEAR CENTRE FOR I-SECTION

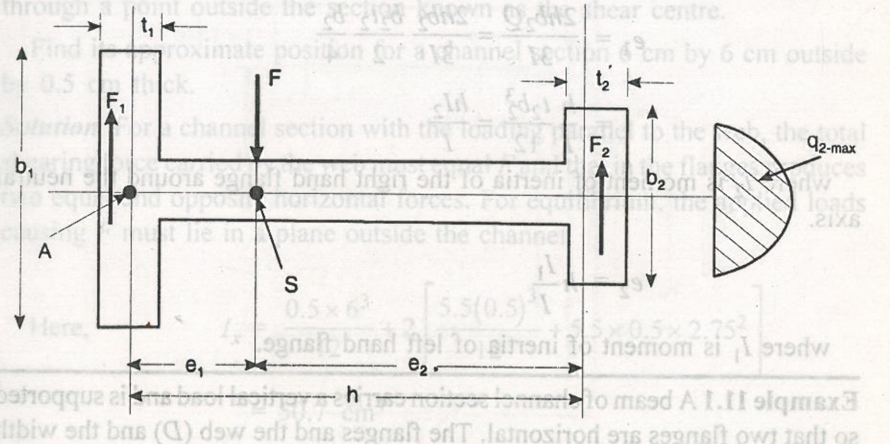


Fig. 11.5

Let us assume an I-section with dimensions shown in Fig. 11.5. This cross section has a horizontal axis of symmetry and the shear centre is located on it, where it is located remains to be found.

Applied force F causes bending and shear stresses only in the flanges. Let the shear force resisted by the left flange of the beam be F_1, and by the right flange, F_2.

For equilibrium

$$F_1 + F_2 = F$$

Likewise, to have no twist of the section,

From $\quad \Sigma M_A = 0$

$$Fe_1 = F_2 h$$

$$\Rightarrow \quad Fe_2 = F_1 h$$

Thus only F_2 remains to be determined to solve the problem. This may be done by noting that the right flange is actually an ordinary rectangular beam. The shear stress in such a beam is distributed parabolically.

Since the area of a parabola is 2/3 of the base times the maximum altitude

$$F_2 = \frac{2}{3} b_2 (q_2)_{max}$$

Since $V = F$

$$(q_2)_{max} = \frac{VQ}{I} = \frac{FQ}{I},$$

where Q is the statical moment of the upper half of the right hand flange and I is the moment of inertia of the whole section. Hence

$$Fe = F_2 h = \frac{2}{3} b_2 (q_2)_{max} h = \frac{\frac{2}{3} h b_2 FQ}{I}$$

$$e_1 = \frac{2hb_2Q}{3I} = \frac{2hb_2}{3I} \frac{b_2 t_2}{2} \frac{b_2}{4}$$

$$= \frac{h}{I} \frac{t_2 b_2^3}{12} = \frac{hI_2}{I}$$

where I_2 is moment of inertia of the right hand flange around the neutral axis.

$$e_2 = h \frac{I_1}{I}$$

where I_1 is moment of inertia of left hand flange.

Example 11.1 A beam of channel section carries a vertical load and is supported so that two flanges are horizontal. The flanges and the web (D) and the width of the flanges (B). Show that the shear centre is at a distance $3B^2/6B + D$ from the web.

Solution.

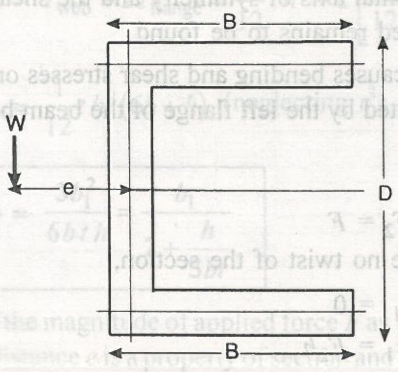


Fig. 11.6

In article 11.3, we have proved that

$$e = \frac{b}{2 + \frac{h}{3b}}$$

Replacing b_1 by B and h by D, we get

$$e = \frac{B}{2 + \frac{D}{3B}}$$

$$= \frac{3B^2}{6B + D}$$

Example 11.2 Explain, why a single channel section with its web vertical subjected to vertical loading as beam, will be in torsion unless the load is applied through a point outside the section known as the shear centre.

Find its approximate position for a channel section 6 cm by 6 cm outside by 0.5 cm thick.

Solution. For a channel section with the loading parallel to the web, the total shearing force carried by the web must equal F and that in the flanges produces two equal and opposite horizontal forces. For equilibrium, the applied loads causing F must lie in a plane outside the channel.

Here, $$I_x = \frac{0.5 \times 6^3}{12} + 2\left[\frac{5.5(0.5)^3}{12} + 5.5 \times 0.5 \times 2.75^2\right]$$

$$= 50.7 \text{ cm}^4$$

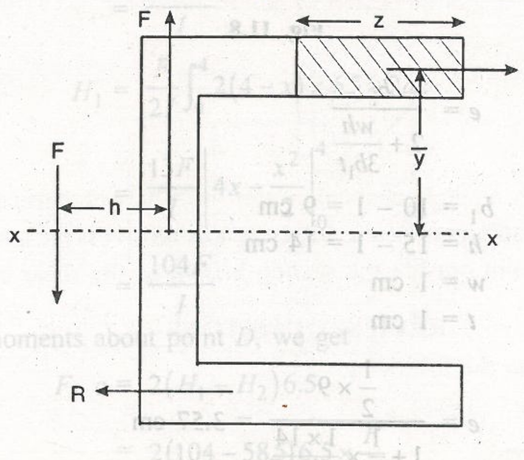


Fig. 11.7

$$\tau = \frac{FA\bar{y}}{It} = \frac{F(zt)2.75}{It} = 0.0543\ Fz$$

$$\text{force } R = \int \tau t\, dz$$

$$= \frac{0.0543}{2} F \left[\frac{z^2}{2}\right]_2^{5.75} = 0.448\ F$$

For equilibrium $Fh = R \times 5.5$

$$\Rightarrow \quad h = \frac{0.448F \times 5.5}{F} = 2.47 \text{ cm}$$

Example 11.3 Determine the shear centre of the channel section shown in Fig. 11.8.

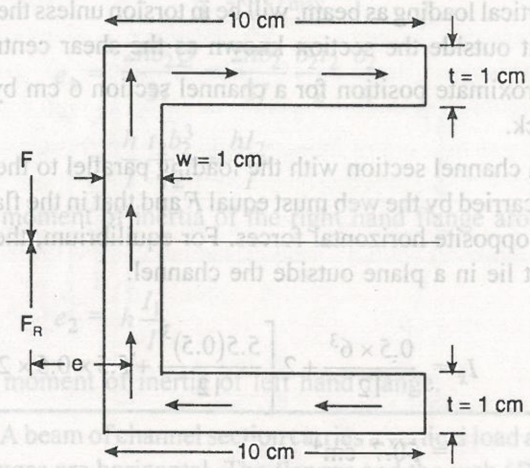


Fig. 11.8

Solution.

$$e = \frac{b_1}{2 + \dfrac{wh}{3b_1 t}}$$

Here

$$b_1 = 10 - 1 = 9 \text{ cm}$$
$$h = 15 - 1 = 14 \text{ cm}$$
$$w = 1 \text{ cm}$$
$$t = 1 \text{ cm}$$

$$\therefore \quad e = \frac{\dfrac{1}{2} \times 9}{1 + \dfrac{1}{6} \times \dfrac{1 \times 14}{9 \times 1}} = 3.57 \text{ cm}$$

Example 11.4 Locate the shear centre of the cross section shown in Fig. 11.9.

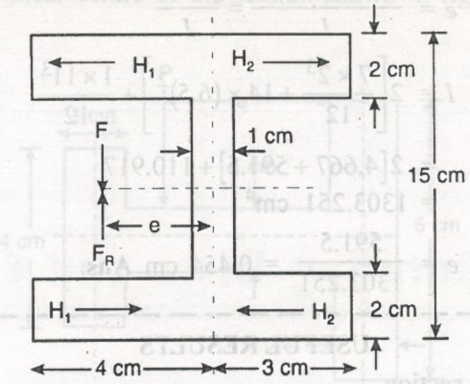


Fig. 11.9

Solution.

$$H_2 = \int \tau dA$$

$$= \int \frac{FA\bar{y}}{It} dA = \frac{F}{It} \int A\bar{y} dA$$

$$= \frac{F}{It} \int_0^3 2(3 - x)(6.5) \times 2dx$$

$$= \frac{F}{2I} \int_0^3 26(3 - x)\, dx$$

$$= \frac{13F}{I} \left| 3x - \frac{x^2}{2} \right|_0^3$$

$$= \frac{58.5F}{I}$$

$$H_1 = \frac{F}{2I} \int_0^4 2(4 - x) \times 6.5 \times 2dx$$

$$= \frac{13F}{I} \left| 4x - \frac{x^2}{2} \right|_0^4$$

$$= \frac{104F}{I}$$

Taking moments about point D, we get

$$F_R\, e = 2(H_1 - H_2)6.5$$

$$= 2(104 - 58.5)6.5 \times \frac{F}{I}$$

Now $F_R = F$

$$\Rightarrow \qquad e = \frac{2 \times 45.5 \times 6.5}{I} = \frac{591.5}{I}$$

$$I = 2\left[\frac{7 \times 2^3}{12} + 14 \times (6.5)^2\right] + \frac{1 \times 11^3}{12}$$

$$= 2[4.667 + 591.5] + 110.917$$

$$= 1303.251 \text{ cm}^4$$

$$\therefore \quad e = \frac{591.5}{1303.251} = 0.454 \text{ cm } \textbf{Ans.}$$

USEFUL RESULTS

1. For channel section

$$e = \frac{b_1}{2 + \dfrac{h}{3b_1}} = \frac{b_1}{2 + \dfrac{wh}{3b_1 t}}$$

2. For unequal *I*-section

$$e_1 = \frac{hI_2}{I}, \quad e_2 = \frac{hI_1}{I}$$

EXPECTED DERIVATIONS/EXPLANATIONS

1. A channel section has a web h deep and w thickness and flanges b_1 wide and t unit thick. Used as a horizontal cantilever with the web in a vertical plane, it carries an end load W. Determine the position of W relative to the web in order that the cantilever shall not be subjected to torsion.

$$\left[\textbf{Hint : } \text{Derive } e = \frac{\frac{1}{2} b_1}{\left(1 + \frac{1}{6}\frac{wh}{b_1 t}\right)}\right]$$

2. Explain, why a single channel section with its web vertical subjected to vertical loading as a beam, will be in tension unless the load is applied through a point outside the section known as the shear centre.

REVIEW QUESITONS

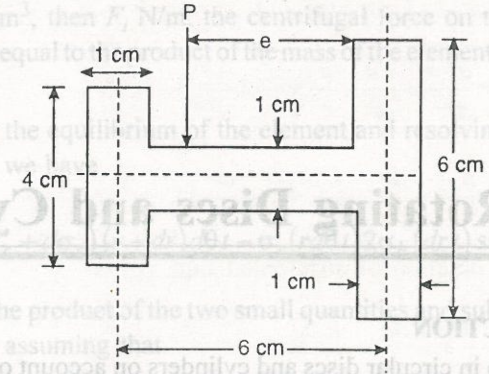
Write short notes on the following

(*i*) Shear Centre

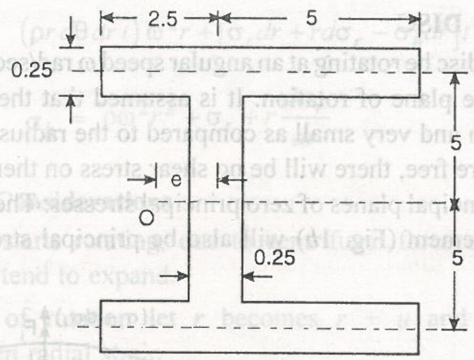
(*ii*) Flexural Axis

NUMERICAL PROBLEMS

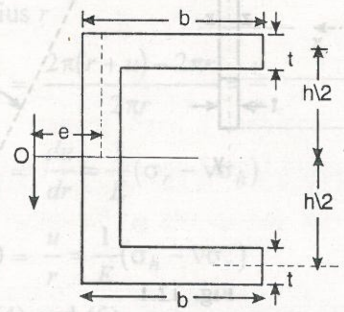
1. Locate the shear centre of the section shown in figure. **[Ans.** 1.37 cm]



2. Locate the shear centre *O* for the unbalanced *I*-section shown in figure, for simple bending in the plane of the web. **[Ans.** e = 1.02 cm]



3. Calculate the distance e from the plane of the web to the shear centre *O* of the channel section shown in Figure below.



Given h = 23.59 cm, b = 7.64 cm, t = 1.41 cm, I = 3828.4 cm^4

[Ans. e = 2.99 cm]

12

Rotating Discs and Cylinders

12.1 INTRODUCTION

Stresses are set up in circular discs and cylinders on account of rotation about their axis of symmetry. The analysis of the stresses set up in a rotating cylinder or circular disc can be made on the basis of certain simplified assumptions.

12.2 ROTATING DISC

Let a thin circular disc be rotating at an angular speed ω rad/sec about its central axis, normal to the plane of rotation. It is assumed that the thickness 't' of the disc is uniform and very small as compared to the radius. Since the faces normal to z-axis are free, there will be no shear stress on them. The flat faces for disc will be principal planes of zero principal stresses. The radial and hoop stresses on any element (Fig. 1b) will also be principal stresses.

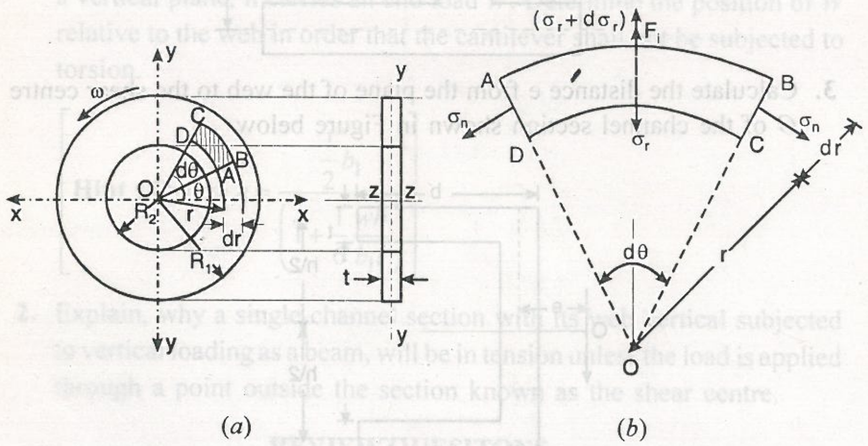


Fig. 12.1

In Fig. 1(b) free body diagram of an element $ABCD$ is shown, such that the radius of the face DC is r and that of AB $(r + dr)$ and the element is bounded

within radial lines AO and BO subtending a small angle $d\theta$ at centre O. The hoop and radial stresses at radius r are represented by σ_h and σ_r N/m^2. Let the radial stress at radius $(r + dr)$ be $(\sigma_r + d\sigma_r)$. Let ρ be the density of the material in kg/m^3, then F_i N/m, the centrifugal force on the element due to rotation will be equal to the product of the mass of the element and the centripetal acceleration.

Considering the equilibrium of the element and resolving the forces along the radial line, we have

$$F_i + (\sigma_r + d\sigma_r)(r + dr)d\theta\, t - \sigma_r (rd\theta\, t) 2\sigma_h (dr\, t) \sin\frac{d\theta}{2} = 0$$

Neglecting the product of the two small quantities and substituting the value of F_i and also assuming that

$$\sin\frac{d\theta}{2} \approx \frac{d\theta}{2}, \text{ we get}$$

$$(\rho\, d\theta\, dr\, t)\,\omega^2 r + [\sigma_r dr + r d\sigma_r - \sigma_h dr]\, t\, d\theta = 0$$

$$\Rightarrow \quad \sigma_h = \rho\omega^2 r^2 + \sigma_r + r\frac{d\sigma_r}{dr} \quad ...(1)$$

12.2.1 Strain Considerations

When the disc starts rotating, due to centrifugal force on each and every element, it will tend to expand.

On account of rotation let r becomes $r + u$ and $r + dr$ becomes $r + dr + du$, then radial strain

$$\varepsilon_r = \frac{(dr + du) - dr}{dr} = \frac{du}{dr} \quad ...(2)$$

Hoop strain at radius r

$$\varepsilon_h = \frac{2\pi(r + u) - 2\pi r}{2\pi r} = \frac{u}{r} \quad ...(3)$$

Now $$\varepsilon_r = \frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_h) \quad ...(4)$$

and $$\varepsilon_h = \frac{u}{r} = \frac{1}{E}(\sigma_h - \nu\sigma_r) \quad ...(5)$$

Solving the Eqn. (4) and (5),

$$\sigma_r = \frac{E}{1 - \nu^2}\left(\frac{\nu u}{r} + \frac{du}{dr}\right) \quad ...(6)$$

$$\sigma_h = \frac{E}{1-\nu^2}\left(\frac{u}{r}+\frac{\nu du}{dr}\right) \qquad ...(7)$$

Putting Eqn. (6) and (7) in Eqn. (1)

$$r\frac{d^2u}{dr^2}+\frac{du}{dr}-\frac{u}{r} = -\rho\frac{\omega^2}{E}\left(1-\nu^2\right)r^2$$

or

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2} = -\rho\frac{\omega^2}{E}\left(1-\nu^2\right)r \qquad ...(8)$$

Complementary function of differential Eqn. (8) is

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2} = 0$$

or

$$\frac{d^2u}{dr^2}+\frac{d}{dr}\left(\frac{u}{r}\right) = 0$$

which on integration gives

$$\frac{du}{dr}+\frac{u}{r} = 2A \qquad ...(9)$$

where $2A$ is a constant of integration.

Equation (9) can be rewritten as

$$r\frac{du}{dr}+u = 2Ar$$

$$\Rightarrow \quad \frac{d}{dr}(ur) = 2Ar$$

On integration, we get $\quad ur = Ar^2 + B$

$$\Rightarrow \quad \frac{u}{r} = A+\frac{B}{r^2} \qquad ...(10)$$

where B is another constant from Eqn. (9) and (10), we get

$$\frac{du}{dr} = A-\frac{B}{r^2} \qquad ...(11)$$

For finding the particular integral of Eqn. (8) assume

$$u = cr^3 \qquad ...(12)$$

where c is a constant.

or

$$\frac{u}{r} = cr^2$$

By differentiating Eqn. (12)

$$\frac{du}{dr} = 3cr^2$$

$$\frac{d^2u}{dr^2} = 6cr \qquad ...(13)$$

Substituting Eqn. (12) through Eqn. (13) in Eqn. (8)

$$\Rightarrow \quad 6cr+\frac{1}{r}3cr^2-cr = \rho\frac{\omega^2}{E}\left(1-\nu^2\right)r$$

or

$$c = -\frac{\rho\omega^2}{8E}\left(1-\nu^2\right)$$

Thus from equation the complete solution is

$$\frac{u}{r} = A+\frac{B}{r^2}-\frac{\rho\omega^2}{8E}\left(1-\nu^2\right)r^2 \qquad ...(14)$$

$$\frac{du}{dr} = A-\frac{B}{r^2}-\frac{3\rho\omega^2}{8E}\left(1-\nu^2\right)r^2 \qquad ...(15)$$

Putting these Eqn. (14) and (15) in Eqns. (6) and (7), we get the radial and hoop stresses as

$$\sigma_r = \frac{E}{\left(1-\nu^2\right)}\left[(1+\nu)A-(1-\nu)\frac{B}{r^2}-(3+\nu)\frac{\rho\omega^2}{8E}\left(1-\nu^2\right)r^2\right] \qquad ...(16)$$

$$\sigma_h = \frac{E}{\left(1-\nu^2\right)}\left[(1+\nu)A+(1-\nu)\frac{B}{r^2}-(1+3\nu)\frac{\rho\omega^2}{8E}\left(1-\nu^2\right)r^2\right] \qquad ...(17)$$

The values of the constants A and B will depend upon the end conditions.

12.3 HOLLOW DISC (DISC WITH A CENTRAL HOLE)

Let R_1 be the outer radius of the disc and R_2 the radius of the central hole. The radial stresses at these radii will be zero. These conditions will help to find the values of constants of integration A and B.

From Eqn. (16)

$$(1+\nu)A-(1-\nu)\frac{B}{R_1^2} = (3+\nu)\frac{\rho\omega^2}{8E}\left(1-\nu^2\right)R_1^2 \qquad ...(18)$$

and $$(1+\nu)A-(1-\nu)\frac{B}{R_2^2} = (3+\nu)\frac{\rho\omega^2}{8E}\left(1-\nu^2\right)R_2^2 \qquad ...(19)$$

Solving the Eqn. (18) and (19), we get

$$A = (3+\nu)(1-\nu)\frac{\rho\omega^2}{8E}\left(R_1^2+R_2^2\right) \qquad ...(20)$$

$$B = (3+\nu)(1+\nu)\frac{\rho\omega^2}{8E}\left(R_1^2 R_2^2\right) \qquad ...(21)$$

Putting these values in Eqn. (16) and (17), we get

$$\sigma_r = \frac{\rho\omega^2}{8}(3+\nu)\left[R_1^2+R_2^2-\frac{R_1^2R_2^2}{r^2}-r^2\right] \qquad ...(22)$$

$$\sigma_h = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_1^2+R_2^2\right)+\frac{R_1^2R_2^2}{r^2}-(1+3\nu)r^2\right] \qquad ...(23)$$

(*i*) For maximum value of σ_r,

$$\frac{d\sigma_r}{dr} = 0$$

$$\Rightarrow \quad \frac{\rho\omega^2}{8}(3+\nu)\left[\frac{2R_1^2R_2^2}{r^3}-2r\right] = 0$$

$$\Rightarrow \quad r = \sqrt{R_1R_2} \qquad ...(24)$$

$$(\sigma_r)_{max} = \frac{\rho\omega^2}{8}(3+\nu)(R_1-R_2)^2$$

(*ii*) Maximum value of σ_h will occur at $r = R_2$

$$(\sigma_h)_{max} = \frac{\rho\omega^2}{4}\left[(3+\nu)R_1^2+(1-\nu)R_2^2\right]$$

12.4 SOLID DISC

In this case,

At $r = R_1$, $\sigma_r = 0$

and at $r = 0$, $u = 0$

By Eqn. (14), we have

$$\frac{u}{r} = A+\frac{B}{r^2}-\frac{\rho\omega^2}{8E}\left(1-\nu^2\right)r^2$$

Eqn. (14) with condition at $r = 0$, $u = 0$ gives $B = 0$.

With the help of first condition (at $r = R_1$, $\sigma_r = 0$), we have from Eqn. (16) for a solid disc

$$\left.\begin{aligned} A &= \frac{\rho\omega^2}{8E}(3+\nu)(1-\nu)R_1^2 \\ B &= 0 \end{aligned}\right\} \qquad ...(25)$$

Thus from Eqns. (25), (16) and (17), we have

$$\sigma_r = \frac{\rho\omega^2}{8}(3+\nu)\left(R_1^2-r^2\right) \qquad ...(26)$$

$$\sigma_h = \frac{\rho\omega^2}{8}(3+\nu)R_1^2-(1+3\nu)r^2 \qquad ...(27)$$

Now, σ_r will be maximum at centre

$$\Rightarrow \quad (\sigma_r)_{max} = \frac{\rho\omega^2}{8}(3+\nu)R_1^2 \qquad ...(28)$$

σ_h will be maximum at $r = 0$

$$\Rightarrow \quad (\sigma_h)_{max} = \frac{\rho\omega^2}{8}(3+\nu)R_1^2 \qquad ...(29)$$

12.5 DISC OF UNIFORM STRENGTH

A disc of uniform strength is the one in which the values of radial and circumferential (hoop) stresses are equal in magnitude at all points in the disc, hence

$$\sigma_h = \sigma_r = \sigma = \text{constant}$$

Consider the equilibrium of the element *ABCD* of the disc shown in Fig. 12.2. Let t be the thickness of the disc at radius r and $t+\Delta t$ at radius $r+\Delta r$, outward radial force acting on face *BC*

$$= \sigma(t+\Delta t)(r+\Delta r)\Delta\theta$$

$$= \sigma(tr+r\Delta t+t\Delta r)\Delta\theta$$

Centrifugal force acting on the element ABCD is

$$= \rho(r\Delta\theta\Delta r\, t)\omega^2 r$$

Inward radial force acting on face $AD = \sigma t r \Delta\theta$

Inward radial force due to component of forces acting on faces AB and CD

$$= \sigma \, t \, \Delta r \Delta\theta$$

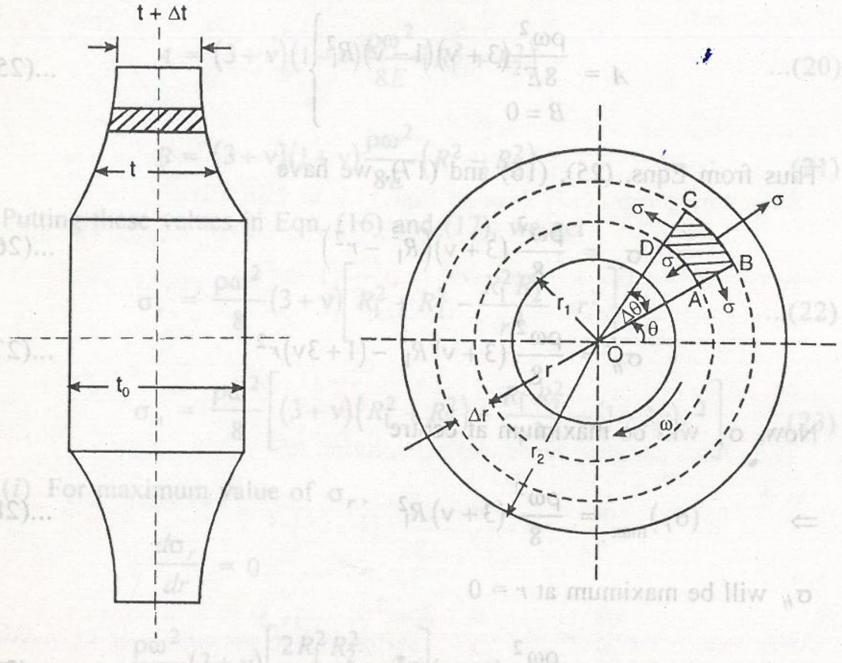


Fig. 12.2

For equilibrium of the element,

Total inward radial force = Total outward radial force

$$\sigma t r \Delta\theta + \sigma t \Delta r \Delta\theta = \sigma(tr + r\Delta t + t\Delta r)\Delta\theta + \rho(r\Delta\theta \Delta r t)\omega^2 r$$

$$\therefore \quad \sigma r \Delta t \Delta\theta + \rho \Delta\theta \Delta r t \omega^2 r^2 = 0$$

$$\frac{\Delta t}{t} = -\rho \frac{\omega^2}{\sigma} r \Delta r$$

$$\Rightarrow \quad \frac{dt}{t} = -\rho \frac{\omega^2}{\sigma} r dr$$

Integrating we get

$$\ln t = \rho \frac{\omega^2}{\sigma} \frac{r^2}{2} + \ln A$$

where $\ln A$ is a constant of integration.

$$\ln \frac{t}{A} = -\rho\left(\frac{\omega^2 r^2}{2\sigma}\right)$$

$$\therefore \quad t = A e^{-\rho\left(\frac{\omega^2 r^2}{2\sigma}\right)}$$

Let $t = t_0$ at $r = r_1$, then

$$t_0 = A e^{-\rho\left(\frac{\omega^2 r_1^2}{2\sigma}\right)}$$

$$\therefore \quad A = t_0 e^{\rho\left(\frac{\omega^2 r_1^2}{2\sigma}\right)}$$

$$\therefore \quad \boxed{t = t_0 \, e^{-\rho \frac{\omega^2}{2\sigma}(r^2 - r_1^2)}}$$

which gives the thickness of disc at any radius.

12.6 ROTATING CYLINDER

Stresses are set up in a circular cylinder on account of rotation about its axis of symmetry.

Consider a circular cylinder of inside radius r_1 and outside r_2 rotating at speed ω (Fig. 12.3).

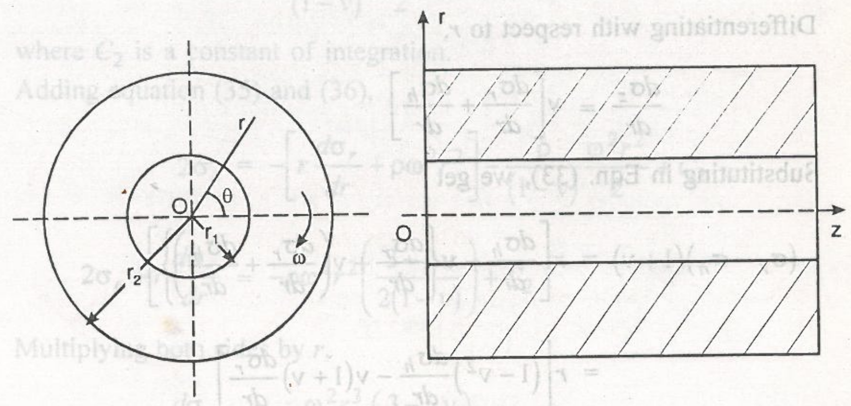


Fig. 12.3

Assume that plane sections of the cylinder remain plane during rotation.

The axial strain along the z-axis will be independent of the radius r of the cylinder and will be constant.

$$\text{Radial strain } \varepsilon_r = \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_z)] = \frac{du}{dr} \qquad \ldots(30)$$

$$\text{Hoop strain } \varepsilon_h = \frac{1}{E}[\sigma_h - \nu(\sigma_r + \sigma_z)] = \frac{u}{r} \qquad \ldots(31)$$

Axial strain $\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_h)\right]$ ...(32)

From Eqn. (31), we have

$$Eu = r\left[\sigma_h - \nu(\sigma_r + \sigma_z)\right]$$

Differentiating with respect to r,

$$E\frac{du}{dr} = \sigma_h - \nu(\sigma_r + \sigma_z) + r\left[\frac{d\sigma_h}{dr} - \nu\left(\frac{d\sigma_h}{dr} - \frac{d\sigma_z}{dr}\right)\right]$$

$$= \sigma_r - \nu(\sigma_h + \sigma_z)$$

By Eqn. (30),

$$(\sigma_r - \sigma_h)(1+\nu) = r\left[\frac{d\sigma_h}{dr} - \nu\left(\frac{d\sigma_r}{dr} - \frac{d\sigma_z}{dr}\right)\right] \quad ...(33)$$

From Eqn. (32),

$$E\varepsilon_z = \sigma_z - \nu(\sigma_r + \sigma_h) = \text{constant} = C_1$$

$$\therefore \quad \sigma_z = C_1 + \nu(\sigma_r + \sigma_h)$$

Differentiating with respect to r,

$$\frac{d\sigma_z}{dr} = \nu\left[\frac{d\sigma_r}{dr} + \frac{d\sigma_h}{dr}\right]$$

Substituting in Eqn. (33), we get

$$(\sigma_r - \sigma_h)(1+\nu) = r\left[\frac{d\sigma_h}{dr} - \nu\left\{\frac{d\sigma_r}{dr} + \nu\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_h}{dr}\right)\right\}\right]$$

$$= r\left[(1-\nu^2)\frac{d\sigma_h}{dr} - \nu(1+\nu)\frac{d\sigma_r}{dr}\right]$$

$$\sigma_r - \sigma_h = r\left[(1-\nu)\frac{d\sigma_h}{dr} - \nu\frac{d\sigma_r}{dr}\right] \quad ...(34)$$

Also considering the equilibrium of an element of the cylinder between angular positions θ and $\theta + d\theta$ and radii r and $r + dr$, we can get as in the case of rotating disc,

$$\sigma_r - \sigma_h = -\left(r\frac{d\sigma_r}{dr} + \rho\omega^2 r^2\right) \quad ...(35)$$

Comparing equations (34) and (35)

$$r\left[(1-\nu)\frac{d\sigma_h}{dr} - \nu\frac{d\sigma_r}{dr}\right] = -r\left[\frac{d\sigma_r}{dr} + \rho\omega^2 r\right]$$

$$(1-\nu)\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr} = -\left(\frac{d\sigma_r}{dr} + \rho\omega^2 r\right)$$

$$(1-\nu)\frac{d\sigma_r}{dr} + (1-\nu)\frac{d\sigma_h}{dr} = -\rho\omega^2 r$$

$$(1-\nu)\left[\frac{d\sigma_r}{dr} + \frac{d\sigma_h}{dr}\right] = -\rho\omega^2 r$$

$$\frac{d\sigma_r}{dr} + \frac{d\sigma_h}{dr} = -\frac{\rho}{(1-\nu)}\omega^2 r$$

$$\frac{d}{dr}(\sigma_r + \sigma_h) = -\frac{\rho}{(1-\nu)}\omega^2 r$$

Integrating,

$$\sigma_r + \sigma_h = -\frac{\rho}{(1-\nu)}\frac{\omega^2 r^2}{2} + C_2 \quad ...(36)$$

where C_2 is a constant of integration.

Adding equation (35) and (36),

$$2\sigma_r = -\left[r\frac{d\sigma_r}{dr} + \rho\omega^2 r^2\right] - \frac{\rho}{(1-\nu)}\frac{\omega^2 r^2}{2} + C_2$$

$$2\sigma_r + r\frac{d\sigma_r}{dr} = -\rho\omega^2 r^2\left(\frac{3-2\nu}{2(1-\nu)}\right) + C_2$$

Multiplying both sides by r,

$$2r\sigma_r + r^2\frac{d\sigma_r}{dr} = -\rho\frac{\omega^2 r^3}{2}\left(\frac{3-2\nu}{1-\nu}\right) + rC_2$$

$$\Rightarrow \quad \frac{d}{dr}(r^2\sigma_r) = -\rho\frac{\omega^2 r^3}{2}\left(\frac{3-2\nu}{1-\nu}\right) + rC_2$$

Integrating,

$$r^2\sigma_r = -\rho\frac{\omega^2 r^4}{8}\left(\frac{3-2\nu}{1-\nu}\right) + \frac{r^2}{2}C_2 + C_3$$

where C_3 is constant of integration.

$$\sigma_r = -\rho\frac{\omega^2 r^2}{8}\left(\frac{3-2\nu}{1-\nu}\right)+\frac{C_2}{2}+\frac{C_3}{r^2} \quad ...(37)$$

Substituting in equation

$$\sigma_h = -\rho\frac{\omega^2 r^2}{8}\left(\frac{1+2\nu}{1-\nu}\right)+\frac{C_2}{2}-\frac{C_3}{r^2} \quad ...(38)$$

Equations (37) and (38) are the governing equations for a rotating cylinder.

12.7 SOLID CYLINDER

From Equation (37) and (38)

$$\sigma_r = \frac{C_2}{2}-\frac{C_3}{r^2}-\rho\frac{\omega^2 r^2}{8}\left(\frac{3+2\nu}{1-\nu}\right)$$

$$\sigma_h = \frac{C_2}{2}-\frac{C_3}{r^2}-\rho\frac{\omega^2 r^2}{8}\left(\frac{1+2\nu}{1-\nu}\right)$$

Constant C_3 must be zero, since the stress remains finite at $r = 0$.

$$\therefore \quad \sigma_r = \frac{C_2}{2}-\frac{1}{8}\left(\frac{3+2\nu}{1-\nu}\right)\rho\omega^2 r^2$$

$$\sigma_h = \frac{C_2}{2}-\frac{1}{8}\left(\frac{1+2\nu}{1-\nu}\right)\rho\omega^2 r^2$$

For a solid cylinder with a free surface at $r = r_2$.

$$\sigma_r = 0$$

$$\therefore \quad \frac{C_2}{2} = \frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2 r_2^2$$

$$\therefore \quad \sigma_r = \frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2\left[r_2^2-r^2\right] \quad ...(39)$$

$$\therefore \quad \sigma_h = \frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2\left[r_2^2-\left(\frac{1+2\nu}{3-2\nu}\right)r^2\right] \quad ...(40)$$

The maximum stresses occur at the centre of the cylinder.

At centre, $r = 0$

$$\therefore \quad (\sigma_r)_{max} = (\sigma_h)_{max} = \frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2 r_2^2 \quad ...(41)$$

12.8 HOLLOW CYLINDER

From equation (37)

$$\sigma_r = \frac{C_2}{2}+\frac{C_3}{r^2}-\frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2 r^2$$

$\sigma_r = 0$ at $r = r_1$ and $r = r_2$

$$\therefore \quad 0 = \frac{C_2}{2}+\frac{C_3}{r_1^2}-\frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2 r_1^2$$

$$0 = \frac{C_2}{2}+\frac{C_3}{r_2^2}-\frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2 r_2^2$$

Solving for C_2 and C_3,

$$\frac{C_2}{2} = \frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2\left(r_1^2+r_2^2\right)$$

$$C_3 = -\frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2 r_1^2 r_2^2$$

$$\therefore \quad \sigma_r = \frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2\left(r_1^2+r_2^2-\frac{r_1^2 r_2^2}{r^2}-r^2\right) \quad ...(42)$$

$$\sigma_h = \frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2\left[r_1^2+r_2^2+\frac{r_1^2 r_2^2}{r^2}-\left(\frac{1+2\nu}{3-2\nu}\right)r^2\right] \quad ...(43)$$

σ_h is maximum at $r = r_1$

$$\therefore \quad (\sigma_h)_{max} = \frac{1}{4}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2 r_2^2\left[1+\left(\frac{1-2\nu}{3-2\nu}\right)\frac{r_1^2}{r_2^2}\right] \quad ...(44)$$

Let $\frac{r_1}{r_2} \approx 0$ then

$$(\sigma_h)_{max} = \frac{1}{4}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2 r_2^2 \quad ...(45)$$

Comparing equations (41) and (45), we can observe that $(\sigma_h)_{max}$ in a cylinder with a small hole at the centre is twice that of $(\sigma_h)_{max}$ in a solid cylinder

For σ_r to be maximum,

$$\frac{d\sigma_r}{dr} = 0$$

$$\Rightarrow \qquad -2r + \frac{2r_1^2 r_2^2}{r^2} = 0$$

$$\Rightarrow \qquad r = \sqrt{r_1 r_2} \qquad ...(46)$$

$$\therefore \qquad (\sigma_r)_{max} = \frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2 (r_2 - r_1)^2 \qquad ...(47)$$

Example 12.1 A hollow cylinder of 40 cm external diameter and 20 cm internal diameter is rotating at 3000 rpm. Determine the distribution of radial and hoop stresses in the cylinder. Density of the cylinder material = 7800 kg/m^3, $\nu = 0.3$.

Solution. For hollow cylinder

$$\sigma_r = \frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2\left(r_1^2 + r_2^2 - \frac{r_1^2 r_2^2}{r^2} - r^2\right)$$

$$= \frac{1}{8}\left(\frac{3-0.6}{1-0.3}\right) \times 7800 \times \left(\frac{2\pi \times 3000}{60}\right)^2 \left[10^2 + 20^2 - \frac{100 \times 400}{r^2} - r^2\right] \times 10^{-4}$$

$$= 0.329927 \times 10^{-4}\left(500 - \frac{40000}{r^2} - r^2\right)$$

$(\sigma_r)_{max}$ at $r = \sqrt{r_1 r_2}$

$$\Rightarrow \qquad r = \sqrt{10 \times 20} = 14.142 \text{ cm}$$

r cm	10	14.142	15	20
σ_r MPa	0	3.299	3.207	0

$$\sigma_h = \frac{1}{8}\left(\frac{3-2\nu}{1-\nu}\right)\rho\omega^2\left[r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \left(\frac{1+2\nu}{3-2\nu}\right)r^2\right]$$

$$= 0.329927 \times 10^{-4}\left[500 + \frac{40000}{r^2} - 0.6667\,r^2\right]$$

r cm	10	15	20
σ_h MPa	27.493	17.41	10.997

The variation of stresses is shown below :

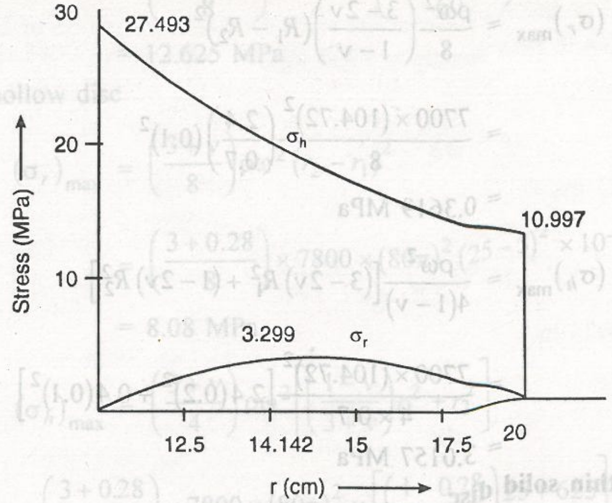


Fig. 12.4

Example 12.2 A turbine rotor, 0.4 m external diameter and 0.2 m internal diameter is revolving at 1000 rpm. Taking the weight of rotor as 7700 kg/m^3 and Poisson's ratio 0.3, find the maximum hoop and radial stresses assuming

(*i*) rotor to be a thin disc

(*ii*) rotor to be a long cylinder

(*iii*) rotor to be a solid disc

(*iv*) rotor to be a solid cylinder

Solution. (*i*) **For thin disc**

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1000}{60} = 104.72 \text{ rad/sec}$$

$$(\sigma_r)_{max} = \frac{\rho\omega^2}{8}(3+\nu)(R_1 - R_2)^2$$

$$= \frac{7700 \times (104.72)^2}{8}(3.3)(0.1)^3 \text{ N/m}^2$$

$$= 0.3475 \text{ MPa}$$

$$(\sigma_h)_{max} = \frac{\rho\omega^2}{4}\left[(3+\nu)R_1^2 + (1-\nu)R_2^2\right]$$

$$= \frac{700 \times (104.72)^2}{4}\left[3.3(0.2)^2 + 0.7(0.1)^2\right] \text{ N/m}^2$$

$$= 2.93 \text{ MPa}$$

(*ii*) **For long cylinder**

$$(\sigma_r)_{max} = \frac{\rho\omega^2}{8}\left(\frac{3-2\nu}{1-\nu}\right)(R_1 - R_2)^2$$

$$= \frac{7700\times(104.72)^2}{8}\left(\frac{2.4}{0.7}\right)(0.1)^2$$

$$= 0.3619 \text{ MPa}$$

$$(\sigma_h)_{max} = \frac{\rho\omega^2}{4(1-\nu)}\left[(3-2\nu)R_1^2 + (1-2\nu)R_2^2\right]$$

$$= \frac{7700\times(104.72)^2}{4\times 0.7}\left[2.4(0.2)^2 + 0.4(0.1)^2\right] \text{ N/m}^2$$

$$= 3.0157 \text{ MPa}$$

(*iii*) **For thin solid disc**

$$(\sigma_r)_{max} = (\sigma_h)_{max} = \frac{\rho\omega^2}{8}(3+\nu)R_2^2$$

$$= \frac{7700\times(104.72)^2}{8}(3.3)(0.2)^2 \text{ N/m}^2$$

$$= 1.39 \text{ MPa}$$

(*iv*) **For long solid cylinder**

$$(\sigma_r)_{max} = (\sigma_h)_{max} = \frac{\rho\omega^2}{8}\left(\frac{3-2\nu}{1-\nu}\right)R_2^2$$

$$= \frac{7700\times(104.72)^2}{8}$$

$$= \left(\frac{2.4}{0.7}\right)(0.2)^2 \text{ N/m}^2$$

$$= 1.4475 \text{ MPa}$$

Example 12.3 A disc of 50 cm diameter and uniform thickness is rotating at 2400 rpm. Determine the maximum stress induced in the disc. If a hole of 10 cm diameter is drilled at the centre of the disc, determine the maximum intensities of radial and hoop stresses induced. Take $\nu = 0.28$, $\rho = 7800$ kg/m^3.

Solution. For the solid disc

$$(\sigma_r)_{max} = (\sigma_h)_{max} = \left(\frac{3+\nu}{8}\right)\rho\omega^2 r_2^2$$

$$= \left(\frac{3+0.28}{8}\right)\times 7800\times\left(\frac{2\pi\times 2400}{60}\right)^2\times 25^2\times 10^{-4}$$

$$= 12.625 \text{ MPa}$$

For the hollow disc

$$(\sigma_r)_{max} = \left(\frac{3+\nu}{8}\right)\rho\omega^2(r_2 - r_1)^2$$

$$= \left(\frac{3+0.28}{8}\right)\times 7800\times(80\pi)^2(25-5)^2\times 10^{-4}$$

$$= 8.08 \text{ MPa}$$

$$(\sigma_h)_{max} = \left(\frac{3+\nu}{4}\right)\rho\omega^2\left[\left(\frac{1-\nu}{3+\nu}\right)r_1^2 + r_2^2\right]$$

$$= \left(\frac{3+0.28}{4}\right)\times 7800\times(80\pi)^2\times\left[\left(\frac{1-0.28}{3+0.28}\right)25 + 625\right]\times 10^{-4}$$

$$= 25.472 \text{ MPa}$$

Example 12.4 A thin uniform steel disc of 25 cm diameter with a central hole of 5 cm diameter, runs at 10000 rpm. Calculate the maximum principal stresses and the maximum shearing stress in the disc. $\nu = 0.3$, density = 7700 kg/m^3.

Solution. The maximum principal stress for thin disc

$$\sigma_h = \frac{\sigma\omega^2}{4}\left[(1-\nu)R_1^2 + (3+\nu)R_2^2\right]$$

$$= \frac{7700}{4}\left(\frac{10000\times 2\pi}{60}\right)^2(0.7\times 0.025^2 + 3.3\times 0.125^2) \text{ N/m}^2$$

$$= 110 \text{ N/mm}^2$$

The maximum shearing stress at any radius is given by

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}(\sigma_h - \sigma_r)$$

$$= \frac{\rho\omega^2}{8}\left[(3+\nu)\frac{R_1^2 R_2^2}{r^2} + (1-\nu)r^2\right]$$

The greatest stress difference will occur at $r = R_1$

$$\therefore \tau_{max} = \frac{7700}{8}\left(\frac{10000\times 2\pi}{60}\right)^2\left(3.3\times\frac{0.025\times 0.125^2}{0.025^2} + 0.7\times 0.025^2\right) \text{ N/m}^2$$

$$= 55 \text{ N/mm}^2$$

Example 12.5 A solid rotor of a turbine is 0.6 m in diameter at the blade ring. It is keyed to a 50 mm diameter shaft. If the minimum thickness is 9.5 mm, what should be the thickness at the shaft for a uniform stress of 200 N/mm^2 at 10000 rpm ? Take density = 7700 kg/m^3.

Solution. $$t = Ae^{-\frac{\rho r^2\omega^2}{2\sigma}}$$

At $r = 0.3$ m

$$t = 9.5 = Ae^{-\frac{\rho\omega^2 \times 0.09}{2\sigma}}$$

At $r = 0.025$ m

$$t = Ae^{-\frac{\rho\omega^2 \times 0.0006}{2\sigma}}$$

$$= 9.5e^{-\frac{\rho\omega^2 \times 0.0894}{2\sigma}}$$

$$= 9.5e^{7700\left(\frac{1000\pi}{30}\right)^2 \frac{0.0894}{2\times 200\times 10^6}}$$

$$= 9.5\ e^{1.89}$$

$$= 63 \text{ mm}$$

Example 12.6 A turbine rotor is 15 cm diameter below the blade ring and 2 cm thick. The turbine is running at 36000 rpm. The allowable stress is 150 MPa. What is the thickness of the rotor at a radius of 5 cm, and at the centre. Assume uniform strength. $\rho = 7800$ kg/m^3.

Solution. $$t = t_0\, e^{-\frac{\rho\omega^2}{2\sigma}\left(r^2 - r_1^2\right)}$$

At $r = 5$ cm

$$t = 2e^{-\frac{7800}{2\times150\times10^6}\times\left(\frac{2\pi\times36000}{60}\right)^2(25-56.25)\times10^{-4}}$$

$$= 2e^{1.15474} = 2 \times 3.1732 = 6.3464 \text{ cm}$$

At $r = 0$

$$t = 2e^{-\frac{7800}{3\times150\times10^6}\times\left(\frac{2\pi\times36000}{60}\right)^2(-56.25)\times10^{-4}}$$

$$= 2e^{2.07854} = 2 \times 7.9928 = 15.985 \text{ cm}$$

Example 12.7 A grinding wheel is 300 mm diameter with the bore at the centre 25 mm diameter. If the thickness of the wheel at the outer radius is 25 mm, what should be the thickness at the bore diameter for a uniform allowable stress of 10 MPa at 2800 rpm ? Take density of the wheel material as 2700 kg/m^3.

Solution. $$\omega = \frac{2\pi}{60}\times 2800 = 293.215 \text{ rad/sec}$$

$$\sigma = 10^7 \text{ N/m}^2$$

$$\rho = 2700 \text{ kg/m}^3$$

$$r_1 = 0.15 \text{ m},\ r_2 = 0.0125,\ t_1 = 0.025 \text{ m},\ t_2 = ?$$

$$t_1 = Ae^{-\frac{\rho\omega r_1^2}{2\sigma}}$$

$$\Rightarrow \quad 0.025 = Ae^{\left(-\frac{2700\times(293.215)^2\times(0.15)^2}{2\times10^7}\right)}$$

$$\Rightarrow \quad A = 0.025e^{-0.26115} \text{ metre}$$

$$= 32.46 \text{ mm}$$

$$t_2 = Ae^{-\frac{\rho\omega r_2^2}{2\sigma}}$$

$$= 32.46e^{\frac{-2700\times(293.215)^2(0.0125)^2}{2\times10^7}}$$

$$= 32.46e^{-0.0018135}$$

$$= 32.40 \text{ mm}$$

EXPECTED DERIVATIONS

1. For a rotating disc with a central hole, show that the maximum value of radial stress is found at a distance $\sqrt{R_1 R_2}$ from the centre of disc, where R_1 and R_2 are the inner and outer radii of the disc respectively. (UPTU 2001-02)
2. Starting from the basic principles, derive an expression for the thickness of a solid rotor of uniform strength. (UPTU 2001-02)

   Hint : Show that $t = t_0\, e^{-\frac{\rho\omega^2}{2\sigma}\left(r_2^2 - r_1^2\right)}$

3. Prove that the maximum circumferential stress in a rotating disc with a central pin hole is twice the value for a solid disc of the same dimension. (UPTU 2002-03)

USEFUL RESULTS

1. For thin disc

$$(\sigma_r)_{max} = \frac{\rho\omega^2}{8}(3+\nu)(R_1 - R_2)^2$$

$$(\sigma_h)_{max} = \frac{\rho\omega^2}{4}\left[(3+\nu)R_1^2 + (1-\nu)R_2^2\right]$$

2. For long cylinder

$$(\sigma_r)_{max} = \frac{\rho\omega^2}{8}\left(\frac{3-2\nu}{1-\nu}\right)(R_1 - R_2)^2$$

$$(\sigma_h)_{max} = \frac{\rho\omega^2}{4(1-\nu)}\left[(3-2\nu)R_1^2 + (1-2\nu)R_2^2\right]$$

3. For thin solid disc

$$(\sigma_r)_{max} = (\sigma_h)_{max} = \frac{\rho\omega^2}{8}(3+\nu)R_1^2$$

4. For long solid cylinder

$$(\sigma_r)_{max} = (\sigma_h)_{max} = \frac{\rho\omega^2}{8}\left(\frac{3-2\nu}{1-\nu}\right)R_1^2$$

5. For hollow disc

$$(\sigma_r)_{max} = \left(\frac{3+\nu}{8}\right)\rho\omega^2(r_2 - r_1)^2$$

$$(\sigma_h)_{max} = \left(\frac{3+\nu}{4}\right) = \rho\omega^2\left[\left(\frac{1-\nu}{3+\nu}\right)r_1^2 + r_2^2\right]$$

6. $t = Ae^{-\frac{\rho r^2\omega^2}{2\sigma}}$

7. $t = t_0\, e^{-\frac{\rho\omega^2(r^2 - r_1^2)}{2\sigma}}$

NUMERICAL PROBLEMS

1. The rotor of a steam turbine is a solid disc of uniform strength and is 20 cm diameter at the blade ring and 2.5 cm thick at the centre. It is running at a constant speed of 30000 rpm. Calculate the thickness of the rotor at a radius of 5 cm. The material density is 7470 kg/m^3 and the maximum allowable stress in the rotor is 145 MPa.
[Ans. 16.83 cm]

2. A thin uniform disc of 25 cm diameter with a central hole of 5 cm diameter runs at 10000 rpm. Calculate the maximum principal stresses and the maximum shearing stress in the disc. ν = 0.3 and ρ = 7470 kg/m^3. **[Ans.** 33.79, 106.49, 36.35 MPa]

3. A thin solid disc of 75 cm diameter is to rotate at 3000 rpm. The material density is 7600 kg/m^3 and Poisson's ratio is 0.28. Plot the variation of radial and hoop stresses in the disc.
[Ans. 43.25 to zero MPa, 43.25 to 18.98 MPa]

4. A long hollow cylinder is of 20 cm external diameter and is 5 cm thick. It is revolving at a constant speed of 2400 rpm. Calculate the maximum radial and hoop stresses induced in the cylinder. ρ = 7600 kg/m^3, μ = 0.30. **[Ans.** 0.514 MPa, 4.286 MPa]

5. A solid cylinder of 25 cm diameter is rotating at 1500 rpm. Determine the maximum hoop stress induced in the cylinder if its material density is 7800 kg/m^3. Poisson's ratio is 0.28. Also draw the variations of radial and hoop stresses in the cylinder.
[Ans. 1.274 to zero MPa, 1.274 to 0.459 MPa]

6. A disc of turbine rotor is 0.5 m diameter. At the blade ring its thickness is 55 cm. It is keyed to a shaft of 50 mm diameter. If the uniform stress in the rotor disc is limited to 200 MPa at 9000 rpm, find the thickness of the disc at the shaft. Density of the rotor material is 7700 kg/m^3.
[Ans. 158.43 mm]

7. Determine the greatest values of radial and hoop stresses for a rotating disc in which the outer and inner radii are 0.3m and 0.15 m. The angular speed is 150 rad/sec. Take Poisson's ratio as 0.304 and density 7700 kg/m^3. **[Ans.** 1.6 MPa, 13.6 MPa]

13

Springs

13.1 INTRODUCTION

Springs are elastic bodies or resilient members which are used to absorb energy and to release it as and when required.

Various types of springs can be designed for different purposes and places, but depending upon the type of resilience, springs may be broadly divided into two categories,

(*i*) Bending springs

(*ii*) Torsion springs.

The types of springs, which are subjected to bending only (and the resilience occurs due to this) are called bending springs. Examples : Laminated springs and leaf springs.

The types of springs, which are subjected to a torsion (and resilience occurs due to this) are called torsion springs. Example : Helical springs.

13.2 HELICAL SPRING

A Helical spring is a piece of wire coiled in the form of helix. If the slope of the helix of the coil is so small, that the bending effects can be neglected, then the spring is called a close-coiled spring. In such a spring only torsional shear stresses are introduced.

If the slope of the helix of the coil is quite appreciable, then both bending as well as torsional shear stresses are introduced in the spring and such type of spring is called an open-coiled spring.

13.3 CLOSED COIL HELICAL SPRING

A coil spring is formed by bending a wire in the form of a helix. If the coils are in close contact with each other, the spring thus formed is called a closed coil spring. In the case of closed coil spring, the helix angle will be very small. Helix angle is defined as the angle which the centre line of the wire makes with

the plane normal to the axis of the spring. Its value is given by $\tan^{-1}\dfrac{p}{\pi D}$, where p is called the pitch which is the distance between the similar points on the adjacent coils of the spring. D is called the mean coil diameter, which is the diameter of an imaginary cylinder, which contains the centre line of the wire that has been wound to form a spring.

13.4 CLOSED COIL HELICAL SPRING UNDER AXIAL LOAD

Consider a closed coil helical spring as shown in Figure 13.1(*a*) under the action of axial load.

Let W = axial load

D = mean coil diameter

d = diameter of spring wire

δ = axial deflection

G = modulus of rigidity

θ = angular deflection

n = number of active coils

τ = maximum shearing stress induced

Torque on the spring acting about the axis of the spring

$$T = \frac{WD}{2}$$

At any radius x from the centre O of the wire, the shearing stress is

$$= \frac{2x}{d}\cdot\tau$$

The torque dT taken up by a ring of width dx at a radius x will be [See Fig. 13.1(*b*)]

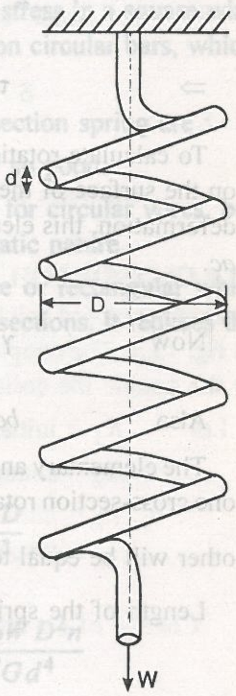


Fig. 13.1(*a*)

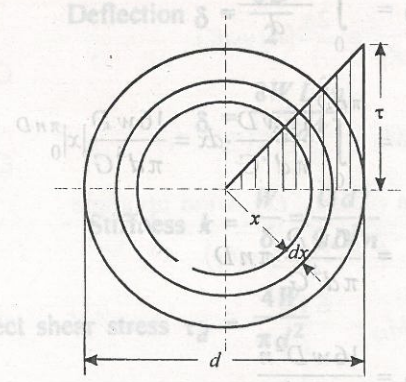


Fig. 13.1(*b*)

$$dT = (2\pi x\cdot dx)\cdot\left(\frac{2x}{d}\tau\right)\cdot x = 4\pi x^3 \tau \frac{dx}{d}$$

Total torque, $T = \int_0^{d/2} 4\pi x^3 \tau \frac{dx}{d}$

$$\frac{WD}{2} = \frac{4\pi\tau}{d}\left[\frac{x^4}{4}\right]_0^{d/2} = \frac{4\pi\tau}{d}\times\frac{d^4}{64} = \frac{\pi\tau d^3}{16}$$

$$\Rightarrow \qquad \tau = \frac{8WD}{\pi d^3} \qquad \ldots(1)$$

To calculate rotation and deflection of the spring, consider an element *ab* on the surface of the bar and parallel to axis as shown in Figure 13.2. After deformation, this element will rotate through a small angle γ to the position *ac*.

Now $\quad \gamma = \frac{\tau}{G} = \frac{8WD}{\pi d^3 G}$

Also $\quad bc = \gamma\, dx$

The elementary angle $d\theta$ through which one cross-section rotates with respect to the other will be equal to $2\gamma\frac{dx}{d}$

Length of the spring wire

$$l = \pi n D$$

$$\theta = \int_0^{\pi n D} \frac{2\gamma\, dx}{d}$$

$$= \int_0^{\pi n D} \frac{16 w D}{\pi d^4 G}\cdot dx = \frac{16 w D}{\pi d^4 G}\left|x\right|_0^{\pi n D}$$

$$= \frac{16 w D}{\pi d^4 G}\cdot \pi n D$$

$$\theta = \frac{16 w D^2 n}{G d^4} \qquad \ldots(2)$$

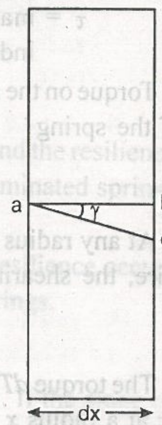


Fig. 13.2

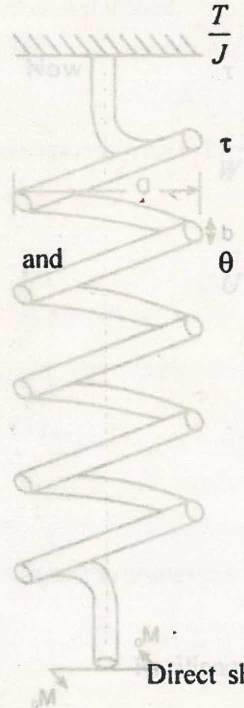
13.5 HELICAL SPRINGS OF NON CIRCULAR WIRE

The use of square or rectangular wire is not recommended for springs, unless space limitations make it necessary. Square or rectangular wire is used to obtain the greatest load capacity in the smallest space, but this means that these springs are highly stressed.

The non circular section of the wire is also used to provide for predetermined altering of the spring rate by grinding off the outside of the coils. For grinding off, the required calculations become very complicated for round wire springs, but are relatively simple for square section wire. The stress in a square wire spring, are based on St. Venant's torsion theory for non circular bars, which we shall discuss in the next chapter.

The main disadvantages of the non-circular wire section spring are :

1. **Quality of the material used for springs is not so good.**
2. **The stress distribution is not as favourable as for circular wires, but this effect is negligible, where loading is of static nature.**
3. **The shape of the wire does not remain square or rectangular while forming helix, this results in trapezoidal cross-sections. It reduces the energy absorbing capacity of the spring.**



$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l}$$

$$\tau = \frac{Tr}{J} = \frac{WD}{2}\times\frac{d}{2}\times\frac{32}{\pi d^4} = \frac{8WD}{\pi d^3}$$

and $\quad \theta = \frac{Tl}{GJ} = \frac{WD\times\pi Dn\times 32}{2G\times\pi\times d^4} = \frac{16WD^2n}{Gd^4}$

Deflection $\delta = \frac{\theta D}{2}$

$$\delta = \frac{8WD^3n}{Gd^4} \qquad \ldots(3)$$

Stiffness $k = \frac{W}{\delta} = \frac{Gd^4}{8D^3n} \qquad \ldots(4)$

Direct shear stress $\tau_d = \frac{4W}{\pi d^2} \qquad \ldots(5)$

Maximum resultant shear stress $= \frac{8WD}{\pi d^3} + \frac{4W}{\pi d^2} \qquad \ldots(6)$

Spring of Square Cross Section Wire

Above results have been derived for spring of circular cross section. However, if the spring is made of square section wire, we have

$$\theta = \frac{410\,T\,l}{N\,b^4} \text{ degrees}$$

$$= \frac{7.11\,T\,l}{N\,b^4} \text{ radians} = \frac{7.11\,W\,R\,l}{N\,b^4} \text{ radians} \quad ...(7)$$

where b = length of each side of square

$$\delta = R\theta = \frac{7.11\,W\,R^2\,l}{N\,b^4} = \frac{44.7\,W\,R^3\,n}{N\,b^4} \quad ...(8)$$

$$\text{Stiffness } k = \frac{W}{\delta} = \frac{N\,b^4}{44.7\,n\,R^3} \quad ...(9)$$

13.6 CLOSED COIL HELICAL SPRING UNDER AXIAL TORQUE

Consider a closed coil helical spring subjected to an axial couple M_0 as shown in fig. 13.3. The couple M_0 produces bending in the coils of the spring. Due to the couple the coils curvature changes.

Let R_1 = initial radius of curvature

R_2 = changed radius of curvature

n_1, n_2 = initial and changed number of coils respectively.

$$\text{Change of curvature} = \frac{1}{R_2} - \frac{1}{R_1} = \frac{M_0}{EI}$$

Now,

$$l = 2\pi R_1 n_1 = 2\pi R_2 n_2$$

$$\frac{1}{R_2} - \frac{1}{R_1} = \frac{2\pi}{l}(n_2 - n_1)$$

$$M_0 = \frac{2\pi E I}{l}(n_2 - n_1)$$

The angle of twist in radians, *i.e.*, wind up angle

$$\phi = 2\pi(n_2 - n_1)$$

$$M_0 = \frac{E I \phi}{l}$$

$$\phi = \frac{M_0 l}{EI} = \frac{2\pi R_1 n_1 M_0}{EI}$$

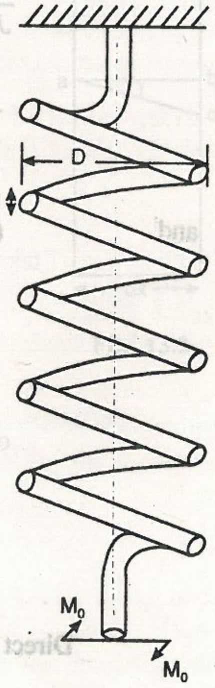


Fig. 13.3

For a spring of round wire of diameter d, number of coils n, and mean coil diameter $D = 2R_1$, we get

$$\phi = \frac{64\pi\,Dn\,M_0}{E\pi d^4}$$

$$= \frac{64\,Dn\,M_0}{Ed^4} \quad ...(10)$$

$$\text{Also bending stress } \sigma_b = \frac{M_0\,y}{I}$$

$$\sigma_b = \frac{M_0 \frac{d}{2}}{\left(\frac{\pi d^4}{64}\right)} = \frac{32}{\pi d^3} M_0 \quad ...(11)$$

13.7 STRAIN ENERGY IN THE SPRING

(*a*) **Under Axial Load :**

$$\text{Strain energy, } U = \frac{1}{2}T\theta = \frac{W}{4} \times D \times \frac{16 W D^2 n}{G d^4} = \frac{4 W^2 D^3 n}{G d^4}$$

Now

$$\tau = \frac{8WD}{\pi d^3}$$

$$W = \frac{\pi d^3 \tau}{8D}$$

$$U = \left(\frac{4 D^3 n}{G d^4}\right)\left(\frac{\pi d^3 \tau}{8D}\right)^2$$

$$= \frac{\tau^2}{G} \times \frac{1}{16} \times D \times n \times d^2 \times \pi^2$$

$$= \frac{\tau^2}{4G} \times (\pi D n)\left(\frac{\pi}{4} d^2\right)$$

$$= \frac{\tau^2}{4G} \times \text{Volume of the spring}$$

$$\text{Resilience of the spring} = \frac{\tau^2}{4G} \quad ...(12)$$

$$\text{Proof Resilience} = \frac{\tau^2_{max}}{4G} \quad ...(13)$$

(*b*) **Under Axial Twist :**

Strain energy $U = \frac{1}{2} M_0 \phi$

$$U = \frac{1}{2} M_0 \times \frac{64\, M_0\, Dn}{E d^4} = \frac{32\, M_0^2\, Dn}{E d^4}$$

Now $\quad M_0 = \frac{\pi d^3}{32} \sigma_b$

$$U = \frac{32\, Dn}{E d^4} \left(\frac{\pi d^3}{32} \sigma_b \right)^2$$

$$= \frac{\sigma_b^2}{8E} \left(\frac{\pi d^2}{4} \right) (\pi\, Dn)$$

$$= \frac{\sigma_b^2}{8E} \times \text{Volume of the spring}$$

$$\text{Resilience of spring} = \frac{\sigma_b^2}{8E} \qquad ...(14)$$

$$\text{Proof Resilience} = \frac{(\sigma_b)^2_{max}}{8E} \qquad ...(15)$$

13.8 SPRINGS IN SERIES

When two springs of different stiffness are joined end to end to carry a common load W, they are said to be connected in series.

$$\text{Total deflection, } \frac{W}{k} = \frac{W}{k_1} + \frac{W}{k_2}$$

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\Rightarrow \qquad k = \frac{k_1 k_2}{k_1 + k_2} \qquad ...(16)$$

where k is the combined stiffness.

13.9 SPRINGS IN PARALLEL

When two springs are joined in such a way that they have a common deflection, then they are said to be connected in parallel.

Total load $W = W_1 + W_2$

Now common deflection

$$\delta = \frac{W}{k} = \frac{W_1}{k_1} = \frac{W_2}{k_2}$$

$$W = \frac{W}{k}(k_1 + k_2)$$

or $\quad k = k_1 + k_2 \qquad ...(17)$

13.10 SPRINGS UNDER IMPACT LOAD

Let a weight W falls on to a spring from a height h measured from the uncompressed state of the spring. Let W_1 be the equivalent static load and δ be the compression of the spring under load W_1.

Work done by falling weight $= W(h + \delta)$

Work stored in the spring $= \frac{1}{2} W_1 \delta$

$$W(h + \delta) = \frac{1}{2} W_1 \delta \qquad ...(18)$$

Now $\quad \delta = \frac{8 W_1 D^3 n}{G d^4}$

Thus, we can determine W_1 and then δ can be determined.

13.11 OPEN COILED HELICAL SPRING UNDER AXIAL LOAD

Consider an open-coiled helical spring as shown in Figure 13.4 under the action of an axial load.

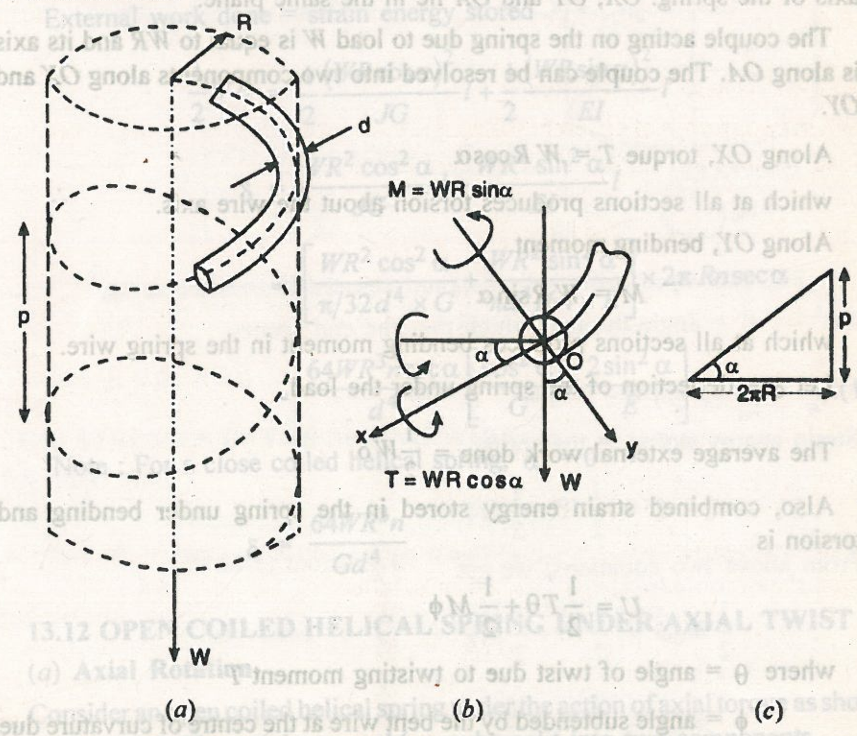


Fig. 13.4 Open Coiled Helical Spring under axial load

Fig. 13.4(a) shows the annular cylinder which contains the axis of the spring wire. The radius of this cylinder will be equal to the mean coil radius of the spring. The axial load W can be considered as a direct load W acting on the spring at the mean radius R, together with a couple WR about OA shown in Fig. 13.4(b).

Let α = constant helix angle which the coils make with planes perpendicular to the axis of the spring.

$D = 2R$ = mean of diameter of the spring coil

n = number of open coils

p = pitch of coils

W = axial load

From Fig. 13.5(c) we have $\tan\alpha = \dfrac{p}{2\pi R}$

Length of spring wire

$$l = \frac{2\pi Rn}{\cos\alpha} = 2\pi Rn\sec\alpha$$

Take an axis OX tangential to the centre line of the open helix at O and OY perpendicular to OX as shown in 13.5(b). Also take OA perpendicular to the axis of the spring. OX, OY and OA lie in the same plane.

The couple acting on the spring due to load W is equal to WR and its axis is along OA. The couple can be resolved into two components along OX and OY.

Along OX, torque $T = WR\cos\alpha$

which at all sections produces torsion about the wire axis.

Along OY, bending moment

$$M = WR\sin\alpha$$

which at all sections produces bending moment in the spring wire.

Let δ = deflection of the spring under the load

The average external work done $= \dfrac{1}{2}W\delta$

Also, combined strain energy stored in the spring under bending and torsion is

$$U = \frac{1}{2}T\theta + \frac{1}{2}M\phi$$

where θ = angle of twist due to twisting moment T

ϕ = angle subtended by the bent wire at the centre of curvature due to bending moment M.

Now $\theta = \dfrac{Tl}{GJ}$

where $J = \dfrac{\pi d^4}{32}$

and $\dfrac{M}{I} = \dfrac{E}{R}$

Also $\dfrac{l}{R} = \phi$

$$R = \frac{l}{\phi}$$

$$\frac{M}{I} = \frac{E\phi}{l}$$

or $\phi = \dfrac{Ml}{EI}$

where $I = \dfrac{\pi d^4}{64}$

For equilibrium of the spring

External work done = strain energy stored

$$\frac{1}{2}W\delta = \frac{1}{2}\frac{(WR\cos\alpha)^2}{JG}l + \frac{1}{2}\frac{(WR\sin\alpha)^2}{EI}l$$

$$\therefore \quad \delta = \frac{WR^2\cos^2\alpha}{JG}l + \frac{WR^2\sin^2\alpha}{EI}l$$

$$= \left[\frac{WR^2\cos^2\alpha}{\pi/32d^4 \times G} + \frac{WR^2\sin^2\alpha}{\pi d^4/64 \times E}\right] \times 2\pi Rn\sec\alpha$$

$$= \frac{64WR^3 n\sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right] \quad ...(19)$$

Note : For a close coiled helical spring, $\alpha = 0$

$$\delta = \frac{64WR^3n}{Gd^4}$$

13.12 OPEN COILED HELICAL SPRING UNDER AXIAL TWIST

(a) Axial Rotation

Consider an open coiled helical spring under the action of axial torque as shown in Fig. 13.5. The axial torque M_0 can be split into two components.

Component along OX

$$T = M_0 \sin\alpha$$

which at all sections produces torsion in the spring wire component along OY,

$$M = M_0 \cos\alpha$$

which at all sections produces bending moment in the spring wire and tends to change the curvature of the coils.

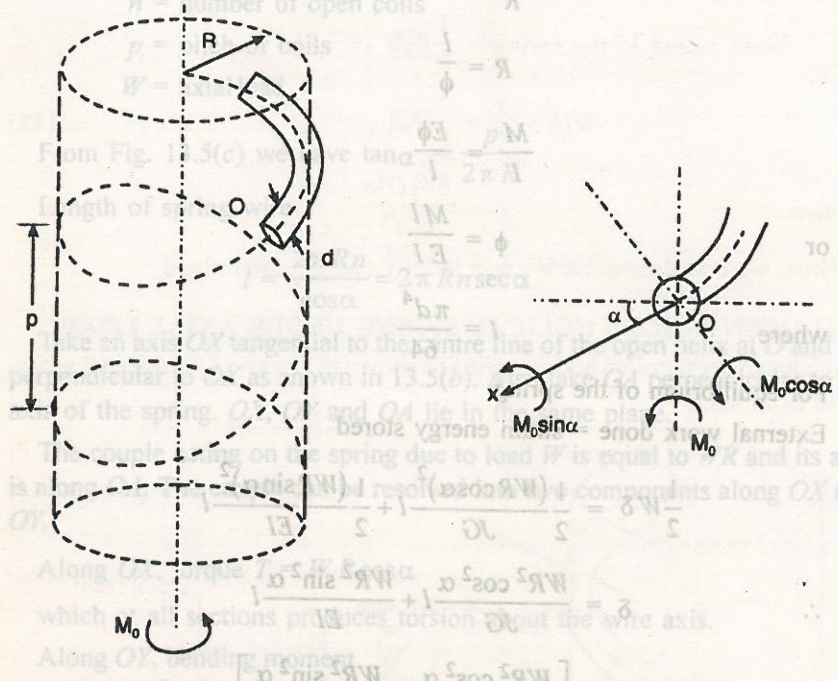


Fig. 13.5 Open coiled helical spring subjected to axial torque M_0

Let β = angle through which the free end rotates

Work done by applied torque $= \frac{1}{2} M_0 \beta$

Strain energy stored in the spring

$$U = \frac{1}{2} T\theta + \frac{1}{2} M\phi$$

From above two equations, we get

$$\frac{1}{2} M_0\beta = \frac{1}{2} T\theta + \frac{1}{2} M\phi$$

$$M_0\beta = \frac{T^2 l}{GJ} + \frac{M^2 l}{EI}$$

$$= \left[\frac{M_0^2 \sin^2\alpha}{G \times \pi d^4/64} + \frac{M_0^2 \cos^2\alpha}{E \times \pi d^4/64}\right] 2\pi Rn \sec\alpha$$

$$\beta = \frac{64 M_0 \; Rn \sec\alpha}{d^4}\left[\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right] \quad ...(20)$$

If $\alpha \approx 0$ then

$$\beta = \frac{64 M_0 Rn}{d^4}\left[0 + \frac{2}{E}\right] = \frac{128 M_0 Rn}{Ed^4}$$

(*b*) **Axial deflection**

Let α = axial deflection

From Fig. 13.6, we get

$$\delta = R\theta\cos\alpha - R\phi\sin\alpha$$

$$= R\frac{Tl}{GJ}\cos\alpha - R\frac{Ml}{EI}\sin\alpha$$

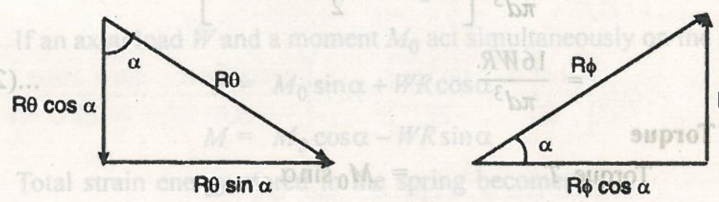


Fig. 13.6

$$= \left[\frac{R \times M_0 \sin\alpha\cos\alpha}{G \times \frac{\pi}{32} d^4} - \frac{R \times M_0 \cos\alpha\sin\alpha}{E \times \frac{\pi}{64} d^4}\right] 2\pi Rn \sec\alpha$$

$$= \frac{64 M_0 R^2 n \sin\alpha}{d^4}\left[\frac{1}{G} - \frac{2}{E}\right] \quad ...(21)$$

when $\alpha \approx 0$, $\delta = 0$

13.13 STRESSES IN THE SPRING WIRE

(*a*) **Axial load**

Torque T $= WR\cos\alpha$

Bending moment M $= WR\sin\alpha$

Shear stress τ $= \frac{16T}{\pi d^3} = \frac{16WR\sin\alpha}{\pi d^3}$

Bending stress σ $= \frac{32M}{\pi d^3} = \frac{32WR\sin\alpha}{\pi d^3}$

Principal stresses are

$$\sigma_{1,2} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}$$

$$= \frac{16WR\sin\alpha}{\pi d^3} \pm \sqrt{\left(\frac{32WR\sin\alpha}{\pi d^3}\right)^2 + 4\left(\frac{16WR\cos\alpha}{\pi d^3}\right)^2}$$

$$= \frac{16WR\sin\alpha}{\pi d^3} \pm \frac{16WR}{\pi d^3}\sqrt{\sin^2\alpha + \cos^2\alpha}$$

$$= \frac{16WR}{\pi d^3}(\sin\alpha \pm 1) \qquad ...(22)$$

Maximum shear stress

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2)$$

$$= \frac{16WR}{\pi d^3}\left[\frac{(\sin\alpha + 1) - (\sin\alpha - 1)}{2}\right]$$

$$= \frac{16WR}{\pi d^3} \qquad ...(23)$$

(*b*) **Axial Torque**

Torque $T = M_0 \sin\alpha$

Bending moment $M = M_0 \cos\alpha$

∴ Shear stress $\tau = \frac{16T}{\pi d^3} = \frac{16 M_0 \sin\alpha}{\pi d^3}$

Bending stress $\sigma = \frac{32M}{\pi d^3} = \frac{32 M_0 \cos\alpha}{\pi d^3}$

Principal stresses are

$$\sigma_{1,2} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}$$

$$= \frac{16 M_0 \cos\alpha}{\pi d^3} \pm \frac{1}{2}\sqrt{\left(\frac{32 M_0 \sin\alpha}{\pi d^3}\right)^2 + 4\left(\frac{16 M_0 \sin\alpha}{\pi d^3}\right)^2}$$

$$= \frac{16 M_0 \cos\alpha}{\pi d^3} \pm \frac{16 M_0}{\pi d^3}\sqrt{\cos^2\alpha + \sin^2\alpha}$$

$$= \frac{16 M_0}{\pi d^3}(\cos\alpha \pm 1) \qquad ...(24)$$

Maximum shear stress

$$\tau_{max} = \frac{16 M_0}{\pi d^3} \qquad ...(25)$$

13.14 AXIAL LOAD AND TWISTING MOMENT ACTING SIMULTANEOUSLY ON CIRCULAR CROSS SECTION

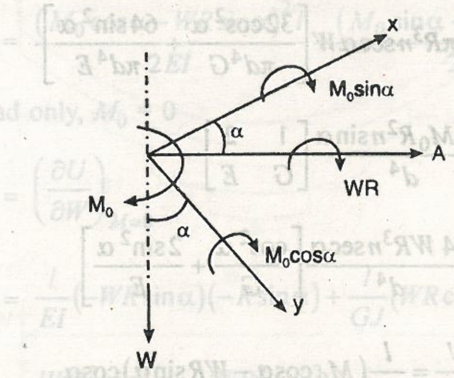


Fig. 13.7 Open coiled helical spring subjected to axial load and twisting moment.

If an axial load W and a moment M_0 act simultaneously on the spring, then

$$T = M_0 \sin\alpha + WR\cos\alpha$$

$$M = M_0 \cos\alpha - WR\sin\alpha$$

Total strain energy stored in the spring becomes

$$U = \frac{M^2 l}{2EI} + \frac{T^2 l}{2GJ}$$

$$= \frac{(M_0\cos\alpha - WR\sin\alpha)^2 l}{2EI} + \frac{(M_0\sin\alpha - WR\cos\alpha)^2 l}{2GJ}$$

Using Castigliano's theorem,

$$\delta = \frac{\partial U}{\partial W} = \frac{l}{EI}(M_0\cos\alpha - WR\sin\alpha)(-R\sin\alpha)$$

$$+ \frac{l}{GJ}(M_0\sin\alpha + WR\cos\alpha)(R\cos\alpha)$$

$$= \frac{lR}{GJ}(M_0\cos\alpha\sin\alpha + WR\cos^2\alpha)$$

$$- \frac{lR}{EI}(M_0\cos\alpha\sin\alpha - WR\sin^2\alpha)$$

$$= l R M_0 \sin\alpha\cos\alpha\left(\frac{1}{GJ} - \frac{1}{EI}\right) + WR^2 l\left(\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right)$$

$$= 2\pi R^2 n \sec\alpha\, M_0 \sin\alpha \cos\alpha \left[\frac{32}{\pi d^4 G} - \frac{64}{\pi d^4 E}\right]$$

$$+ 2\pi R^3 n \sec\alpha\, W \left[\frac{32\cos^2\alpha}{\pi d^4 G} + \frac{64 \sin^2\alpha}{\pi d^4 E}\right]$$

$$= \frac{64\, M_0 R^2 n \sin\alpha}{d^4}\left[\frac{1}{G} - \frac{2}{E}\right]$$

$$+ \frac{64\, W R^3 n \sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right] \qquad ...(26)$$

$$\phi = \frac{\partial U}{\partial M_0} = \frac{l}{EI}(M_0 \cos\alpha - WR\sin\alpha)\cos\alpha$$

$$+ \frac{l}{GJ}(M_0 \sin\alpha - WR\cos\alpha)\sin\alpha$$

$$= lM_0\left[\frac{\cos^2\alpha}{EI} + \frac{\sin^2\alpha}{EI}\right] + lWR\left[\frac{\cos\alpha \sin\alpha}{GJ} - \frac{\sin\alpha \cos\alpha}{EI}\right]$$

$$= \frac{64\, M_0 R n \sec\alpha}{d^4}\left[\frac{2\cos^2\alpha}{E} + \frac{\sin^2\alpha}{G}\right]$$

$$+ \frac{64 W R^2 n \sin\alpha}{d^4}\left[\frac{1}{G} - \frac{2}{E}\right] \qquad ...(27)$$

If the end is fixed against rotation, then

$$\phi = 0$$

and

$$M_0 = \frac{WR\sin\alpha \cos\alpha\left[\frac{2}{E} - \frac{1}{G}\right]}{\left[\frac{2\cos^2\alpha}{E} + \frac{\sin^2\alpha}{G}\right]} \qquad ...(28)$$

13.15 DEFLECTION OF SPRING BY ENERGY METHOD

(Use of Castigliano's theorem)

Let us find the axial deflection of the free end of an open coiled spring with the help of Castigliano's theorem in the following cases :

(*a*) when only an axial load W acts

(*b*) when only an axial moment M acts.

Case (*a*) In article 13.14 we have seen that total strain energy stored in the spring becomes :

$$U = \frac{(M_0 \cos\alpha - WR\sin\alpha)^2 l}{2EI} + \frac{(M_0 \sin\alpha + WR\cos\alpha)^2 l}{2GJ}$$

For axial load only, $M_0 = 0$

$$\therefore \quad \delta = \left(\frac{\partial U}{\partial W}\right)_{M=0}$$

$$= \frac{l}{EI}(-WR\sin\alpha)(-R\sin\alpha) + \frac{l}{GJ}(WR\cos\alpha)(R\cos\alpha)$$

$$= \frac{lWR^2 \sin^2\alpha}{EI} + \frac{lWR^2\cos^2\alpha}{GJ}$$

$$= WR^2 l\left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right]$$

Case (*b*) For axial moment only, $W = 0$

$$\therefore \quad \delta = \left(\frac{\partial U}{\partial W}\right)_{W=0}$$

$$= \frac{l}{EI}(M_0 \cos\alpha)(-R\sin\alpha) + \frac{l}{GJ}(M_0 \sin\alpha)(R\cos\alpha)$$

$$= -\frac{M_0 R l}{EI}\sin\alpha\cos\alpha + \frac{M_0 R l}{GJ}\sin\alpha\cos\alpha$$

$$= \frac{M_0 R l}{2}\sin 2\alpha\left(\frac{1}{GJ} - \frac{1}{EI}\right)$$

13.16 LEAF SPRING

The leaf spring is also called as carriage or laminated spring. This is made by placing circularly bent spring steel strips or plates of same radius one over the other. Each plate is free to slide relative to one over the other. Each plate is free to slide relative to the adjacent plates.

A leaf spring is a beam of uniform strength supported at the centre and loaded at the ends. It consists of a number of overlapping leaves each of the same width and depth but varying in length. Each leaf is shorter than the one above it by a constant amount, called the overlap. Each plate acts as a separate beam as it is free to slide. Each plate has initially the same curvature.

Consider a semi-elliptical leaf spring as shown in Fig. 13.8.

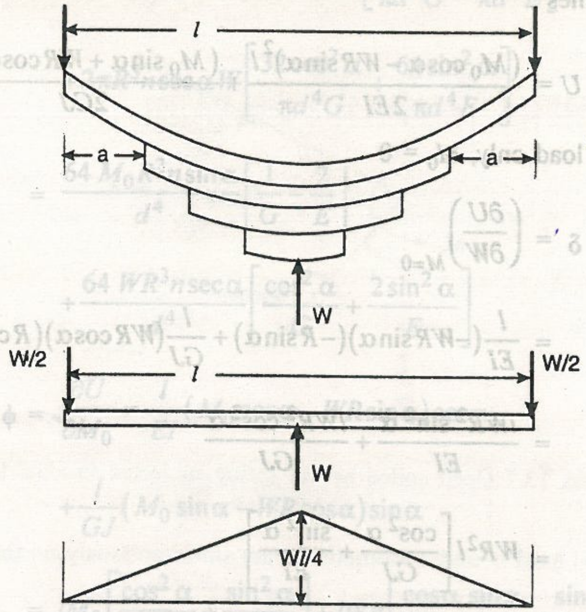


Fig. 13.8 Semi elliptical leaf spring

Let W = Load acting at mid point

n = Number of plates

a = Overlap at each end

l = Length of the spring

b = Width of plate

d = Depth of plate

Then $l = 2an$

$$\therefore \quad a = \frac{l}{2n}$$

Maximum bending moment

$$M_{\max} = \frac{Wl}{4}$$

Bending moment for each plate

$$M = \frac{Wl}{4n}$$

Now bending stress

$$\sigma = \frac{M}{Z}$$

$$\therefore \quad \sigma = \frac{Wl}{4n}\frac{6}{bd^2} = \frac{3}{2}\frac{Wl}{nbd^2} \qquad \ldots(29)$$

$\therefore$ Change of curvature for each plate after loading

$$\frac{1}{R} = \frac{M}{EI} = \frac{Wl}{4n} \times \frac{12}{Ebd^3} = \frac{3Wl}{nEbd^3}$$

Now $$EI\frac{d^2y}{dx^2} = M$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{3Wl}{nEbd^3}$$

$$\therefore \quad y = \int\int \frac{d^2y}{dx^2}\,dx\,dx = \int_0^{l/2}\int_0^{l/2} \frac{3Wl}{nEbd^3}\,dx\,dx$$

$$= \frac{3Wl}{nEbd^3}\int_0^{l/2} x\,dx = \frac{3Wl}{nEbd^3} \times \frac{l^2}{8}$$

$\therefore$ Deflection at the centre

$$\delta = \frac{3}{8}\frac{Wl^3}{nEbd^3} \qquad \ldots(30)$$

Strain energy absorbed

$$U = \frac{M^2}{2EI} \times \text{length of the beam}$$

$$= \frac{\left(\frac{bd^2}{6}\sigma\right)^2}{2E\frac{bd^3}{12}} \times \text{Total length of all the leaves}$$

$$= \frac{\frac{bd^2\sigma^2}{6E}}{6E} \times \text{Total length of all the leaves}$$

$$= \frac{\sigma^2}{6E} \times \text{Volume of spring}$$

$$\therefore \quad \text{Resilience} = \frac{\sigma^2}{6E} \qquad \ldots(31)$$

Quarter Elliptic Leaf Spring

If we substitute $W = 2W$ and $l = 2\,l$ in Eqns. (29) and (30), then we get the expressions for a quarter elliptical spring.

$$\therefore \quad \sigma = \frac{3}{2}\frac{2W\,2l}{nbd^2} = \frac{6Wl}{nbd^2} \qquad \text{...(32)}$$

$$\delta = \frac{3}{8}\frac{2W(2l)^3}{nEbd^3} = \frac{6Wl^3}{nEbd^3} \qquad \text{...(33)}$$

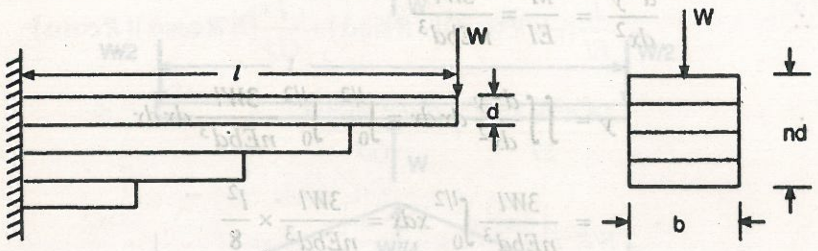


Fig. 13.9 Quarter elliptic leaf spring

Example 13.1 A close coiled helical spring of 10 cm mean diameter is made up of 1 cm diameter rod and has 20 turns. The spring carries an axial load of 200 N. Determine the shearing stress. Taking the value of modulus of rigidity = 8.4×10^4 N/mm², determine the deflection when carrying this load. Also calculate the stiffness of the spring and the frequency of free vibration for a mass hanging from it.

Solution. $\tau = \dfrac{8WD}{\pi d^3}$

$$= \frac{8 \times 200 \times 100}{\pi \times (10)^3} = 50.93 \text{ N/mm}^2 \textbf{ Ans.}$$

$$\delta = \frac{8WD^3 n}{Gd^4}$$

$$= \frac{8 \times 200 \times (100)^3 20}{(8.4 \times 10^4)(10)^4} = 38.095 \text{ mm } \textbf{Ans.}$$

$$\text{Stiffness of spring} = \frac{\text{Load on spring}}{\text{Deflection of spring}}$$

$$= \frac{200}{38.095} = 5.25 \text{ N/mm } \textbf{Ans.}$$

$$\delta = 38.095 = 3.8096 \text{ cm}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{981}{3.8095}} = 2.55 \text{ cycle/sec } \textbf{Ans.}$$

Example 13.2 A close coiled helical spring consisting of 8 coils, each having mean diameter 80 mm and wire diameter 10 mm. The spring is fixed at one end and a twisting moment of 10 Nm is applied axially at the other end in such a way that the spring tends to open. Determine : (*a*) the maximum bending stress produced in the wire (*b*) the angle of twist (*c*) the resilience and (*d*) the number of turns after the application of torque. $E = 2 \times 10^5$ N/mm².

Solution. The maximum bending stress produced in the spring wire

$$\sigma = \frac{My}{I} = \frac{\left(M \times \frac{d}{2}\right)}{\left(\frac{\pi}{64}d^4\right)} = \frac{32M}{\pi d^3} = \frac{32 \times 10 \times 10^3}{\pi \times 10^3}$$

$$= 101.85 \text{ N/mm}^2$$

Let angle of twist = ϕ

$$\phi = \frac{Ml}{EI}, \quad \text{where } l = \text{solid length}$$

$$\Rightarrow \quad \phi = \frac{M \times 2\pi Rn}{E \times \frac{\pi}{64}d^4} = \frac{128MRn}{Ed^4}$$

$$= \frac{128 \times 10 \times 10^3 \times 40 \times 8}{(2 \times 10^5) \times (10)^4} = 0.20 \text{ radian}$$

$$= 1°28' \textbf{ Ans.}$$

Volume of the spring

$$V = \frac{\pi}{4}d^2 \times 2\pi Rn$$

$$= \frac{\pi}{4}10^2 \times 2\pi \times 40 \times 8 = 157914 \text{ mm}^3$$

$$\text{Resilience} = \frac{\sigma^2 V}{8E} = \frac{(101.85)^2 \times 157914}{8 \times (2 \times 10^5)} = 1024 \text{ mm } \textbf{Ans.}$$

$$\phi = 2\pi(n - n')$$

$$\Rightarrow \quad n' = n - \frac{\phi}{2\pi} = 8 - \frac{0.20}{2\pi} = 7.97 \text{ turns } \textbf{Ans.}$$

Example 13.3 The stiffness of a close coiled helical spring is 1.5 N/mm of compression under a maximum load of 60 N. The maximum shearing stress produced in the wire of the spring is 125 N/mm². The solid length of the spring (when the coils are touching) is given as 5 cm. Find : (*i*) diameter of wire (*ii*) mean diameter of the coils and (*iii*) number of coils required. Take $G = 4.5 \times 10^4$ N/mm².

Solution. Stiffness $K = \frac{Gd^4}{8D^3n}$ $(D = 2R)$

$$= \frac{Gd^4}{64R^3n}$$

$$\Rightarrow \quad 1.5 = \frac{(4.5 \times 10^4)d^4}{64R^3n}$$

$$\Rightarrow \quad d^4 = 0.002133\ R^3n \qquad \text{...(1)}$$

$$\tau = \frac{8WD}{\pi d^3}$$

$$\text{or} \quad \tau = \frac{16WR}{\pi d^3}$$

$$\Rightarrow \quad 125 = \frac{16 \times 60 \times R}{\pi d^3}$$

$$\Rightarrow \quad R = 0.40906d^3 \qquad \text{...(2)}$$

By Eqn. (1) and (2)

$$d^4 = 0.00014599 \times d^9n$$

$$\Rightarrow \quad d^5n = \frac{1}{0.000145999} \qquad \text{...(3)}$$

Solid length $= n \times d$

$$\Rightarrow \quad 50 = nd$$

$$\Rightarrow \quad n = \frac{50}{d} \qquad \text{...(4)}$$

By Eqn. (3) and (4)

$$d^5 \times \frac{50}{d} = \frac{1}{0.00014599}$$

$$\Rightarrow \quad d = 3.42 \text{ mm } \textbf{Ans.}$$

Substituting this value in Eqn. (4)

$$n = \frac{50}{3.42}$$

$$= 14.62 \approx 15 \textbf{ Ans.}$$

From Eqn. (2)

$$R = 0.40906 \times (3.42)^3 = 16.36 \text{ mm}$$

$\therefore$ Mean diameter of coil $D = 2R$

$$= 2 \times 16.36$$

$$= 32.72 \text{ mm } \textbf{Ans.}$$

Example 13.4 In a cross-coiled spring, the diameter of each coil is 10 times that of wire of the spring. The maximum shear stress is not to exceed 60 N/mm². Maximum permissible deflection under a load of 400 N is 10 cm. Determine the number of coils, the diameter of the coil and energy stored in the coil. $G = 9 \times 10^4$ N/mm².

Solution. Let diameter of the wire $= d$

$$\therefore \quad D = 10d$$

Load W will cause a twisting moment

$$T = W\frac{D}{2}$$

We know that twisting moment

$$T = \frac{\pi}{16}\tau \times d^3$$

$$\therefore \quad W\frac{D}{2} = \frac{\pi}{16}\tau d^3$$

$$\Rightarrow \quad 400\frac{D}{2} = \frac{\pi}{16} \times 60 \times d^3$$

$$\Rightarrow \quad 400\frac{(10d)}{2} = \frac{\pi}{16} \times 60 \times d^3$$

$$\Rightarrow \quad d = 13.02 \text{ mm}$$

$\therefore$ Diameter of the coil, $D = 10d = 10 \times 13.02 = 130.2$ mm **Ans.**

$$\delta = \frac{8WD^3n}{Gd^4}$$

$$\Rightarrow \quad 100 = \frac{8 \times 400 \times (130.2)^3 \times n}{(9 \times 10^4)(13.02)^4}$$

$$\Rightarrow \quad n = 36.61 \approx 37 \textbf{ Ans.}$$

Let energy stored = U

$$U = \frac{1}{2} \times W \times \delta$$

$$\Rightarrow \quad U = \frac{1}{2} \times 400 \times 100 = 20000 \text{ Nmm } \textbf{Ans.}$$

Example 13.5 A railway wagon weighing 25 kN is moving at a speed of 3 kmph. How many springs each of 24 coils will be required in a buffer stop to absorb the energy of motion during a compression of 200 mm ? The mean diameter of coils is 240 mm and the diameter of steel rod comprising the coils is 20 mm. $G = 0.9 \times 10^5$ N/mm^2, g = 9.8 m/sec^2.

Solution. v = 3 km/hour = 83.4 cm/sec

Kinetic energy of the wagon = $\frac{1}{2} \times mv^2$

where m = mass of wagon = $\frac{W}{g}$

$$\Rightarrow \quad K.E. = \frac{1}{2} \times \frac{25 \times 1000 \times 83.4^2}{9.8} = 0.8863 \times 10^3 \text{ N-m}$$

$$= 0.8863 \times 10^6 \text{ N-mm}$$

If W is the axial load, for each spring for a compression of δ,

$$\delta = \frac{8WD^3n}{Gd^4}$$

$$\Rightarrow \quad W = \frac{Gd^4\delta}{8D^3n} = \frac{0.9 \times 10^5 \times (20)^4 (200)}{8 \times (240)^3 \times 24} = 1085 \text{ N}$$

Energy stored by one spring = $\frac{1}{2} W\delta$

$$= \frac{1}{2} \times 1085 \times 200 = 10.85 \times 10^4 \text{ N-mm}$$

$\therefore$ Number of springs required

$$= \frac{0.8863 \times 10^6}{10.85 \times 10^4} = 8.17 \approx 9 \textbf{ Ans.}$$

Example 13.6 A weight of 200 N is dropped on to a closely coiled helical spring made of 15 mm steel wire coiled to a mean diameter of 150 mm with 24 coils. If the instantaneous compression is 100 mm. Calculate the height of drop. $G = 0.90 \times 10^5$ N/mm^2

Solution. Let h be the height of drop, in mm

W = Gradually applied load in N to produce the same compression

$$\delta = \frac{8WD^3n}{Gd^4}$$

$$\Rightarrow \quad W = \frac{Gd^4\delta}{8D^3n} = \frac{(0.90 \times 10^5)(15)^4 (100)}{8(150)^3 (24)} = 707 \text{ N}$$

Equating the energy supplied by the impact load to the energy stored,

$$P(h + \delta) = \frac{1}{2} W\delta$$

$$\Rightarrow \quad 200(h + 100) = \frac{1}{2} \times 707 \times 100$$

$$\Rightarrow \quad h = 76.8 \text{ mm } \textbf{Ans.}$$

Example 13.7 Two close coiled helical springs wound from the same wire, but with different core radii having equal number of coils, are compressed between rigid plates at their ends. Calculate the maximum shear stress induced in each spring, if the wire diameter is 10 mm and the load applied between the rigid plates is 500 N. The core radii of the springs are 100 mm and 75 mm respectively.

Solution. $n_1 = n_2$, $\quad d$ = 10 mm

W = 500 N

R_1 = Radius of outer spring = 100 mm

R_2 = Radius of inner spring = 75 mm

Let W_1 = Load shared by outer spring

W_2 = Load shared by inner spring

$$\delta_1 = \frac{64W_1R_1^3n_1}{Gd^4} = \frac{64 \times W_1(100)^3 \times n_1}{G(10)^4} = \frac{6400W_1n_1}{G} \quad \ldots(1)$$

$$\text{Similarly } \delta_2 = \frac{64W_2R_2^3n_2}{Gd^4} = \frac{64 \times W_2(75)^3 \times n_2}{G(10)^4} = \frac{2700W_2n_2}{G} \quad \ldots(2)$$

Since the springs are held between two rigid plates, deflections in both the springs must be equal.

Equating Eqn. (1) and (2) gives,

$$W_1 = \frac{27W_2}{64} \quad \ldots(3)$$

$$\text{Also} \quad W_1 + W_2 = 500 \quad \ldots(4)$$

By Eqn. (3) and (4),

$$W_2 = 351.6 \text{ N}$$

$$W_1 = 148.4 \text{ N}$$

From relation of torque for outer spring,

$$W_1 R_1 = \frac{\pi}{16} \tau_1 d^3$$

$$\Rightarrow \quad 148.4 \times 100 = \frac{\pi}{16} \times \tau_1 \times (10)^3$$

$$\Rightarrow \quad \tau_1 = 75.6 \text{ N/mm}^2 \textbf{ Ans.}$$

Similarly, $\tau_2 = \dfrac{351.6 \times 75 \times 16}{\pi(10)^3} = 134.3 \text{ N/mm}^2$ **Ans.**

Example 13.8 An open coiled helical spring consists of 10 coils, each of mean diameter 5 cm, the wire forming the coils being 6 mm diameter, and making a constant angle of 30° with planes perpendicular to the axis of the spring. What load will cause the spring to elongate 1.25 cm and what will be the bending and shearing stresses due to this load ? Calculate the value of axial twist which would cause a bending stress of 56 MPa in the coils. E = 210 GPa and G = 84 GPa.

Solution. $$\delta = \frac{8WD^3 n \sec\alpha}{d^4}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$\Rightarrow \quad 1.25 \times 10^{-2} = \frac{8W \times (0.05)^3 \times 10 \times \sec 30°}{(0.6 \times 10^{-2})^4}\left[\frac{\cos^2 30°}{84 \times 10^9} + \frac{2\sin^2 30°}{210 \times 10^9}\right]$$

$$\Rightarrow \quad W = 124.05 \text{ N}$$

$$\text{Torque } T = \frac{WD}{2}\cos\alpha$$

$$= 124.05 \times \frac{5}{2} \times 10^{-2} \times \cos 30° = 2.6857 \text{ Nm}$$

$$\text{Shear stress } \tau = \frac{16T}{\pi d^3} = \frac{16 \times 2.6857}{\pi \times (0.6 \times 10^{-2})^3} = 63.325 \text{ MPa}$$

$$\text{Bending Moment } M = \frac{WD\sin\alpha}{2}$$

$$= 124.05 \times \frac{5}{2} \times 10^{-2} \sin 30° = 1.5506 \text{ N-m}$$

$$\text{Bending stress } \sigma = \frac{32M}{\pi d^3} = \frac{32 \times 1.5506}{\pi(0.6 \times 10^{-2})^3} = 73.122 \text{ MPa}$$

Let M_0 = axial torque

Bending moment $M = M_0 \cos\alpha$

$$\text{Bending stress } \sigma = \frac{32 M_0 \cos\alpha}{\pi d^3}$$

$$\Rightarrow \quad 56 \times 10^6 = \frac{32 M_0 \cos 30°}{\pi(0.6 \times 10^{-2})^3}$$

$$\Rightarrow \quad M_0 = 1.371 \text{ N-m}$$

Example 13.9 An open coiled helical spring of 5 cm mean diameter is made of steel of 6 mm diameter. Calculate the number of turns required in the spring to give a deflection of 1.2 cm for an axial load of 250 N, if the angle of helix is 30°. Calculate also the rotation of one end of the spring relative to the other if it is subjected to an axial couple of 10 N-m. E = 210 GPa, G = 84 GPa

Solution. $$\delta = \frac{8WD^3 n \sec\alpha}{d^4}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$\Rightarrow \quad 1.2 \times 10^{-2} = \frac{8 \times (250)(0.5)^3 n \sec 30°}{(6 \times 10^{-3})^4}\left(\frac{\cos^2 30°}{84 \times 10^9} + \frac{2\sin^2 30°}{210 \times 10^9}\right)$$

$$\Rightarrow \quad n = 4.76$$

Using Eqn. (20)

$$\beta = \frac{64 M_0 R n \sec\alpha}{d^4}\left(\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right)$$

$$= \frac{64 \times 10 \times (2.5 \times 10^{-2})(4.76)\sec 30°}{(6 \times 10^{-3})^4}\left(\frac{\sin^2 30°}{84 \times 10^9} + \frac{2\cos^2 30°}{210 \times 10^9}\right)$$

$$= 0.687 \text{ radian}$$

Example 13.10 An open coiled helical spring is made having n turns wound to a mean diameter d. The wire diameter is d and the coils make an angle of α with a plane perpendicular to the axis of the coil. Prove that the angle of rotation of free end will be given by

$$\phi = \frac{16WD^2 n \sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

Solution. Total strain energy stored in the spring

$$U = \frac{M^2 l}{2EI} + \frac{T^2 l}{2GJ}$$

Putting $T = M_0 \sin\alpha + WR\cos\alpha$

$M = M_0 \cos\alpha - WR\sin\alpha$

We have, $$U = \frac{(M_0\cos\alpha - WR\sin\alpha)^2 l}{2EI} + \frac{(M_0\sin\alpha + WR\cos\alpha)^2 l}{2GJ}$$

$$\phi = \left(\frac{\partial U}{\partial M_0}\right)_{M_0=0}$$

$$\Rightarrow \quad \phi = \frac{2[-(WR\sin\alpha)]\cos\alpha\, l}{2EI} + \frac{2[WR\cos\alpha]\sin\alpha\, l}{2GJ}$$

$$= WRl\sin\alpha\cos\alpha\left(\frac{1}{GJ} - \frac{1}{EI}\right)$$

Putting $J = \dfrac{\pi d^4}{32}$ and $I = \dfrac{\pi d^4}{64}$

$$\phi = \frac{WRl\sin\alpha\cos\alpha \times 32}{\pi d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \left[\frac{W \times (D/2) \times (\pi Dn/\cos\alpha) \times \sin\alpha\cos\alpha \times 32}{\pi d^4}\right]\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \frac{16WD^2 n\sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

Example 13.11 A leaf spring 100 cm in length is required to carry a central load of 250 kg. If the central deflection is not to exceed 30 mm, determine (*a*) thickness of plates (*b*) number of plates (*c*) the radius to which the plates are to be bent. Bending stress is limited to 2000 kg/cm², $E = 2 \times 10^6$ kg/cm². Width of each plate = 10 times its thickness.

Solution. Let thickness of the plate = t

∴ Width of the plate = $10\,t$

$$\delta = \frac{\sigma L^2}{4Ed}$$

$$\Rightarrow \quad 3 = \frac{2000 \times 100^2}{4 \times 2 \times 10^6 \times t}$$

$$\Rightarrow \quad t = 0.83 \text{ cm} = 8.3 \text{ mm}$$

Let number of plates = n

$$\sigma = \frac{3Wl}{2nbd^2}$$

$$\Rightarrow \quad 2000 = \frac{3 \times 250 \times 100}{2 \times n \times 8.3 \times 0.83^2}$$

$$\Rightarrow \quad n = 3.27$$

$$\approx 4 \text{ number}$$

Radius of the plates

$$R = \frac{Et}{2\sigma}$$

$$\Rightarrow \quad R = \frac{2 \times 10^6 \times 0.83}{2 \times 2000} = 415 \text{ cm}$$

Example 13.12 A leaf spring has 12 plates, each 50 mm wide and 5 mm thick, the longest plate being 600 mm long. The greatest bending stress is not to exceed 180 N/mm² and the central deflection is 15 mm. Estimate the magnitude of the greatest central load that can be applied to the spring. $E = 0.206 \times 10^6$ N/mm².

Solution. (*i*) From deflection consideration

$$\delta = \frac{3Wl^3}{8nEbt^3}$$

$$\Rightarrow \quad W = \frac{8\delta Enbt^3}{3l^3} = \frac{8 \times 15 \times 0.206 \times 10^6 \times 12 \times 50(5)^3}{3 \times (600)^3}$$

$$= 2860 \text{ N}$$

(*ii*) From stress consideration

$$\sigma = \frac{3Wl}{2nbt^2}$$

$$\Rightarrow \quad W = \frac{2\sigma nbt^2}{3l} = \frac{2 \times 180 \times 12 \times 50 \times (15)^2}{3 \times 600}$$

$$= 3000 \text{ N}$$

Since $2860 < 3000$

∴ Allowable load = 2860 N

Example 13.13 A laminated spring, simply supported at the ends and centrally loaded with a span of 75 cm is required to carry a proof load as 7.5 kN and the central deflection is not to exceed 50 mm. The bending stress must not be greater than 400 N/mm^2. Plates are available in multiple of 1 mm for thickness and in multiples of 3 mm for width.

Determine suitable values for thickness, width, number of plates and the radius to which the plates should be formed. Assume the width to be twelve times the thickness. $E = 2 \times 10^5\ N/mm^2$.

Solution. $$\delta = \frac{3Wl^3}{8nEbd^3}$$

$$\Rightarrow \quad nbt^3 = \frac{3Wl^3}{8E\delta} \quad \ldots(1)$$

Also $$\sigma = \frac{3Wl}{2nbd^2}$$

$$\Rightarrow \quad nbt^2 = \frac{3Wl}{2\sigma} \quad \ldots(2)$$

Dividing Eqn. (2) by (1),

$$\Rightarrow \quad t = \frac{l^2\sigma}{4E\delta}$$

Putting the values

$$t = \frac{(750)^2 \times 400}{4 \times (2 \times 10^5)(50)} = 5.63 \text{ mm}$$

Plates will be available in 1 mm, 2 mm, ..., 6 mm etc. thickness

$\therefore$ Nearest available thickness = 6 mm

Given, $b = 12\ t$

$\therefore \quad b = 12 \times 6 = 72$ mm

From Eqn. (2)

$$n = \frac{3Wl}{2\sigma bt^2} = \frac{3 \times 7500 \times 750}{2 \times 400 \times 72 \times (6)^2} = 8.14 \approx 9$$

Let number of plates = 9

$\therefore$ From Eqn. (2), the modified value of bending stress is given by

$$\sigma = \frac{3}{2}\frac{Wl}{nbt^2} = \frac{3 \times 7500 \times 750}{2 \times 9 \times 72 \times (6)^2} = 361.7\ N/mm^2$$

Radius of the plates $R = \dfrac{Et}{2\sigma}$

$$\Rightarrow \quad R = \frac{(2 \times 10^5)(6)}{2 \times 361.7} = 1660 \text{ mm}$$

Example 13.14 A quarter elliptic leaf spring 800 mm long is subjected to a point load of 10 kN. If the bending stress and deflection is not to exceed 320 MPa and 80 mm respectively. Find the suitable size and number of plates required by taking the width as 8 times the thickness. Take $E = 200$ GPa.

Solution. We know that for a quarter elliptic spring,

$$\delta = \frac{6Wl^3}{nEbd^3} \quad \ldots(1)$$

and $$\sigma = \frac{6Wl}{nbd^2} \quad \ldots(2)$$

Dividing (2) by Eqn. (1), we have

Thickness $$d = \frac{\sigma l^2}{E\delta}$$

$$= \frac{320 \times (800)^2}{(200 \times 10^3)(80)}$$

$$= 12.8 \approx 13 \text{ mm}$$

Width of the plate $b = 8t = 8 \times 13 = 104$ mm **Ans.**

From Eqn. (1)

$$320 = \frac{6 \times (10 \times 10^3) \times 800}{n \times 104 \times (13)^2}$$

$\Rightarrow \quad n = 8.5 \approx 9$ **Ans.**

Example 13.15 A quarter elliptic leaf spring has a length of 50 mm and consists of plates each 5 cm wide and 6 mm thick. Find the least number of plates which can be used, if the deflection under a gradually applied load of 2 kN is not to exceed 7 cm.

If instead of being gradually applied the load of 2 kN falls from distance of 6 mm on the undeflected spring, find the maximum deflection and stress produced. $E = 200$ GPa.

Solution. $$\delta = \frac{6Wl^3}{nEbd^3}$$

$$\Rightarrow \quad 0.07 = \frac{6 \times (2 \times 10^3)(0.5)^3}{n \times (200 \times 10^9) \times (5 \times 10^{-2})(6 \times 10^{-3})^3}$$

$$\Rightarrow \quad n = 9.92 \approx 10$$

Let W_e be the equivalent gradually applied load which would produce the same deflection as is caused by the impact load.

$$\therefore \quad \delta_1 = \frac{6W_e l^3}{nEbd^3}$$

$$= \frac{6W_e \times (0.5)^3}{10 \times (200 \times 10^9) \times (5 \times 10^{-2})(6 \times 10^{-3})^3}$$

$$\Rightarrow \quad W_e = 28.8 \times 10^3 \delta_1 \text{ N}$$

Work done by the falling weight on spring

$$= W(h + \delta_1)$$

Strain energy absorbed by spring $= \frac{1}{2} W_e \delta_1$

$$W(h + \delta_1) = \frac{1}{2} W_e \delta_1$$

$$\Rightarrow \quad 2 \times 10^3 (6 \times 10^{-3} + \delta_1) = \frac{1}{2} \times (28.8 \times 10^3 \delta_1) \delta_1$$

$$\Rightarrow \quad \delta_1^2 - 0.139\delta_1 - 0.83 \times 10^{-3} = 0$$

Solving for δ_1, we get,

$$\delta_1 = 144.65 \text{ mm}$$

$$W_e = 28.8 \times 10^3 \times 144.65 \times 10^{-3} = 4.166 \text{ kN}$$

Maximum stress produced

$$\sigma_{max} = \frac{6W_e l}{nbd^2}$$

$$= \frac{6 \times (4.166 \times 10^3) \times 0.5}{10 \times (5 \times 10^{-2}) \times (6 \times 10^{-3})^2}$$

$$= 694.32 \text{ MPa}$$

EXPECTED DERIVATIONS

1. A closely coiled helical spring with D as diameter of the coil and d as diameter of the wire is subjected to an axial load W. Prove that the maximum shear stress produced is $8WD/\pi d^3$.
2. Derive an equation for the deflection of an open coiled helical spring.
3. Deduce an expression for the extension of an open coiled helical spring carrying an axial load W. Take α as the inclination of the coils, d the diameter of the wire and R the mean radius of the coils.
4. Derive from first principles, making usual assumptions the formula for the maximum bending stress and for the central deflection of a leaf spring consisting of n leaves and subjected to a central load.
5. Prove that the deflection of a close-coiled helical spring at the centre due to axial load W is given by $\delta = 64WR^3n/Gd^4$

   All symbols are used in their usual meanings.
6. Find an expression for the strain energy stored by the close-coiled helical spring when subjected to axial load W.
7. An open coiled spring carries an axial load W. Derive expression for displacement and angular twist of the free end.
8. Derive an expression for the axial extension of an open coiled helical spring produced by an axial twisting couple. (UPTU 2001-02)
9. Derive an expression for the axial extension of an open coiled helical spring produced by an axial load. (UPTU 2002-03)
10. Prove that in an open coiled helical spring, subjected to an axial load, the value of the maximum shear stress is the same as in a close-coiled spring of the same dimensions.
11. Prove that the central deflection of the leaf spring (laminated spring) is given by

$$\delta = \frac{3Wl^3}{8nEbd^3}$$

12. An open coiled helical spring is made having n turns wound to a mean diameter D. The wire diameter is d and the coils make an angle of α with a plane perpendicular to the axis of the coil. Prove that the angle of rotation of free end will be given by

$$\phi = \frac{16WD^2 n \sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

USEFUL RESULTS

(a) Closed coil spring

1. $\tau = \dfrac{8WD}{\pi d^3}$ **2.** $\delta = \dfrac{8WD^3 n}{Gd^4}$

3. $K = \dfrac{W}{\delta} = \dfrac{Gd^4}{8D^3 n}$ **4.** Stiffness = $\dfrac{\text{Load on spring}}{\text{Deflection of spring}}$

5. $\theta = \dfrac{16WD^2 n}{Gd^4}$

6. Angle of twist $\phi = 2\pi(n_2 - n_1)$

$$= \frac{Ml}{EI} = \frac{M \times 2\pi Rn}{E(\pi d^4/64)}$$

7. Work done by the falling weight on spring = weight falling $(h+\delta)$

$$\Rightarrow \frac{1}{2}W_1\delta = W(h+\delta)$$

8. Energy stored in the spring = $\dfrac{1}{2}W\delta$

9. Total length of the wire L = Length of one coil × number of coils = $2\pi Rn$

10. Total gap in coils = Gap between two adjacent coils × number of turns

11. Frequency of free vibration $\phi = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{\delta}}$

12. Solid length = length of spring when fully compressed = nd

13. Volume of spring $V = \dfrac{\pi}{4}d^2 \times 2\pi Rn$

14. Resilience = $\dfrac{\tau^2}{4G}$ × Volume of spring

(b) Open Coiled Springs

15. Deflection under axial load or axial extension

$$= \frac{8WD^3 n \sec\alpha}{d^4}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

16. Angle of rotation of free end

$$\phi = \frac{16WD^2 n \sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

17. Torque $T = \dfrac{WD}{2}\cos\alpha$ **18.** $M = \dfrac{WD}{2}\sin\alpha$

19. $\sigma = \dfrac{32M}{\pi d^3}$

20. $\beta = \dfrac{64 M_0 Rn \sec\alpha}{d^4}\left[\dfrac{\sin^2\alpha}{G} + \dfrac{2\cos^2\alpha}{E}\right]$

(c) Semi elliptic leaf spring

21. $\delta = \dfrac{3Wl^3}{8nEbd^3} = \dfrac{\sigma l^2}{4Ed}$

22. $\sigma = \dfrac{3Wl}{2nbd^2}$ **23.** overlap = $\dfrac{l}{2n}$

24. Resilience = $\dfrac{\sigma^2}{6E}$

25. Strain energy absorbed = $\dfrac{\sigma^2}{6E}$ × volume of spring

26. Radius of the plates R = $\dfrac{Et}{2\sigma}$ where t is thickness

(d) Quarter elliptic leaf spring

27. $\delta = \dfrac{6Wl^3}{nEbd^3}$ **28.** $\sigma = \dfrac{6Wl}{nbd^2}$

29. $\dfrac{\sigma}{y} = \dfrac{E}{R}$

REVIEW QUESTIONS

Write short notes on the following :

(i) Laminated spring

(ii) Helical spring

(iii) Leaf spring

(iv) Open coiled helical spring

(v) Closed coil helical spring

(vi) Carriage spring

(vii) Deflection of spring by energy method

(viii) Helical spring of non-circular wire

NUMERICAL PROBLMES

1. A close coiled helical spring is to have a stiffness of 70 kN/m and to exert a force of 2.25 kN. If the mean diameter of the coils is to be 90 mm and the working stress 230 MPa, find the required number of coils and the diameter of the steel rod from which the spring should be made. Take modulus of rigidity as 80 GPa. [Ans : 6.58, 81, 13.08 mm]
2. A closed coil spring is to have a stiffness of 1 kN/m of compression, a maximum load of 50 N and a maximum shearing stress of 120 MPa. The solid length of the spring is to be 45 mm. Find the diameter of the wire, the mean diameter of the coils and the number of coils required. G = 50 GPa. [**Ans.** 0.0034 m, .0436 m, 12.54]
3. An open coiled spring of 125 mm mean diameter has 10 coils of 12 mm diameter wire, at a slope of 30° to the horizontal when the coil axis is vertical. Find the expressions for the longitudinal extension and the rotation for the joint application of an axial load W and an axial torque T. Hence find the axial load and torque necessary to extend the spring 5 mm, if rotation is prevented. E = 200 GPa, G = 80 GPa. [**Ans.** 48.9 N, 0.312 Nm]
4. A laminated spring made of 12 steel plates, is 0.9 m long. The maximum central load is 7.2 kN. If the maximum allowable stress in steel is 230 MPa and maximum deflection is approximately 38 mm, calculate the width and thickness of the plates. E = 200 GPa. [**Ans.** 93.8 mm, 6.13 rad]
5. Deduce an expression for the resilience of a loaded carriage spring, the maximum bending stress is given. A carriage spring 1.35 m long has leaves of 100 × 12.5 mm section. The maximum bending stress is 150 MPa and the spring must absorb 125 J when straightened. Calculated the number of leaves and their initial curvature. E=200 GPa. [**Ans.** 8, 8.43 m]
6. A leaf spring spans 1 m and is supported at each end. It carries two concentrated loads of 180 kg each at points 0.3 m from each end. It is made from leaves 5 cm wide and 6.3 mm thick.

   Design the number and length of the leaves in order that the maximum stress in the material shall not exceed 280 N/mm^2.

   (**Ans.** 6 leaves, lengths 50, 60, 70, 80, 90 and 100 cm)
7. Determine the weight of a close -coiled helical spring to carry a load of 5000 N with a deflection of 5 cm and a maximum shearing stress of 400 N/mm^2. If the number of active coils is 8, determine the wire diameter and mean coil diameter. G = 83000 N/mm^2, ρ = 7700 kg/m^3. (**Ans.** 2 kg, 13.6 mm, 75 mm)

14

Torsion of Non Circular and Hollow Sections

14.1 INTRODUCTION

The ordinary theory of torsion, which you have gone through in III semester is true only for circular sections. For other sections, this theory is not applicable. In developing the theory of torsion of circular shafts we had assumed that the plane sections normal to the axis remain plane even after the application of torque. However, for shafts of non-circular sections, it is no longer possible to prove that the plane normal cross-sections remain plane or they remain undistorted in their own plane.

The detailed analysis of the torsion of non circular sections which includes the warping of sections is beyond the scope of this text. However, we present some of the formulae, without proof, for calculating maximum shear stress and angle of twist for important non-circular sections.

14.2 RECTANGULAR SECTIONS

For rectangular shafts with longer side a and shorter side b, the maximum shear stress when subjected to a torque τ occurs at centre of the longer side and is given by

$$\tau_{max} = \frac{\tau}{k_1 d b^2} \quad ...(1)$$

where k_1 is a constant depending on the ratio d/b.

The angle of twist per unit length is given by

$$\frac{\theta}{l} = \frac{\tau}{k_2 d b^3 G} \quad ...(2)$$

where k_2 is another constant depending on the d/b ratio, Equation (1) and (2) may be approximated by the following equations

$$\tau_{max} = \frac{T}{d b^3}(3d + 1.8b) \quad ...(3)$$

and $$\theta = \frac{42 T l J}{G d^4 b^4} \quad ...(4)$$

where $$J = \frac{bd}{12}\left(b^2 + d^2\right)$$

For narrow rectangular sections, *i.e.* when the rectangular section becomes longer and thinner, the values of constants k_1 and k_2 approach 0.333.

Therefore for all practical purposes when $d/b > 10$,

both k_1 and k_2 may be taken to be equal to 1/3.

Equation (1) and (2) reduce to

$$\tau_{max} = \frac{3T}{d b^2} \quad ...(5)$$

$$\frac{\theta}{l} = \frac{3T}{d b^3 G} \quad ...(6)$$

14.3 EQUILATERAL TRIANGULAR SECTION

For an equilateral triangular section shaft of side $2a$, the maximum shear stress occurs at the middle of each side and is given by

$$\tau_{max} = \frac{2.5T}{a^3} \quad ...(7)$$

and the angle of twist per unit length

$$\frac{\theta}{l} = \frac{5T}{\sqrt{3}\, a^4 G} \quad ...(8)$$

14.4 ELLIPTICAL SECTIONS

For an elliptical shaft of major axis $2a$ and minor axis $2b$, the maximum shear stress occurs at the end of the minor axis,

i.e., $y = b$ and is given by

$$\tau_{max} = \frac{2T}{\pi a b^2} \quad ...(9)$$

The angle of twist per unit length is given by

$$\frac{\theta}{l} = \frac{\left(a^2 + b^2\right)T}{\pi a^3 b^3 G} \quad ...(10)$$

14.5 SAINT-VENANT'S THEORY OF TWISTING OF NON-CIRCULAR SHAFTS

Saint Venant made following assumptions for the development of this theory:

(*i*) The stresses are within elastic limit and Hooke's law holds good.

(*ii*) Twist of any cross-section is proportional to its distance from the fixed end, which is taken as the reference plane.

(*iii*) Plane cross-sections turn bodily about the centre without distortion in the plane. However, a cross-section warps in the longitudinal direction and this warping is different for different points of the section. It is further assumed that all cross-sections warp in the same manner.

(*iv*) There is no longitudinal load acting on the shaft.

14.6 TORSION OF THIN TUBULAR SECTION

Let us consider a closed tube of small thickness acted upon by a torque T in a transverse plane (Fig. 14.1).

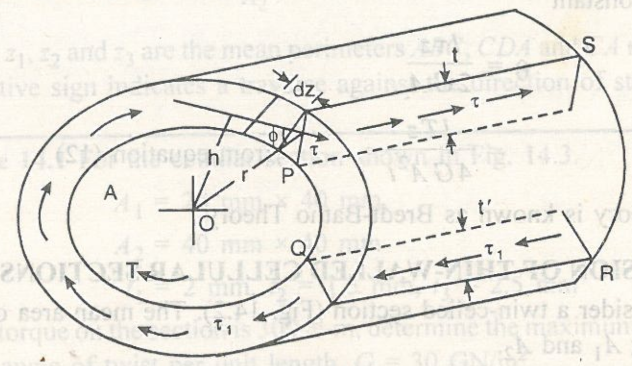


Fig. 14.1

If it is assumed that the shear stress τ at point P where the thickness is t is constant across the tube wall, then if τ' is the shear stress at Q and t' the thickness, then from the equilibrium of the complimentary shear stresses on PS and QR

$$\tau t = \tau' t' = k \quad ...(11)$$

If dz is an element round the circumference, then the force on this element will be $(\tau t \cdot dz)$

Taking moments about O,

$$T = \int \tau \cdot t \cdot dz \cdot r \sin\phi$$

$$= k \int h \cdot dz$$

where h is the perpendicular distance from O on to τ, hence

$$T = 2kA \quad ...(12)$$

where A = Area enclosed by the mean circumference.

The strain energy of length l of tube is

$$U = \int \frac{\tau^2}{2G} \cdot l t \, dz$$

$$= \frac{kl}{2G} \int \tau \cdot dz \qquad \text{from equation (11)}$$

But $U = \frac{1}{2} T\theta$, where θ is the angle of twist,

Hence $$\theta = \left(\frac{kl}{TG}\right) \int \tau \, dz$$

$$= \frac{l}{2GA} \int \tau \, dz \qquad \text{from equation (12)}$$

If t is constant

$$\theta = \frac{l\tau z}{2GA} \qquad \text{...(13)}$$

$$= \frac{lTz}{4GA^2 t} \qquad \text{from equation (12)}$$

This theory is known as Bredt-Batho Theory.

14.7 TORSION OF THIN-WALLED CELLULAR SECTIONS

Let us consider a twin-celled section (Fig. 14.2). The mean area of the two cells being A_1 and A_2.

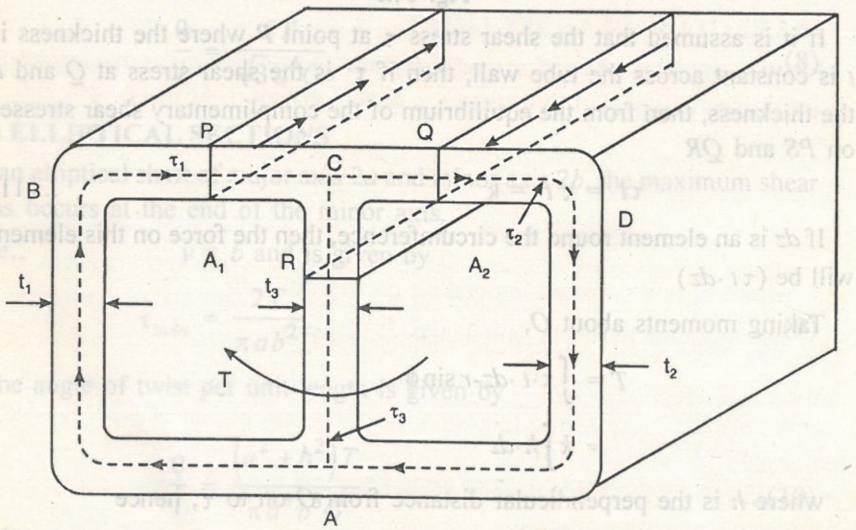


Fig. 14.2

If the length ABC is of uniform thickness t_1 and stress τ_1, CDA of thickness t_2 and stress τ_2, and CA of thickness t_3 and stress τ_3.

Then from the equilibrium of complimentary shear stresses on a longitudinal section through PQR

$$\tau_1 t_1 = \tau_2 t_2 + \tau_3 t_3 \qquad \text{...(13)}$$

The total torque on the section by using equation (12) and adding for the two cells,

$$\tau = 2(\tau_1 t_1 A_1 + \tau_2 t_2 A_2) \qquad \text{...(14)}$$

Applying equation (13) to each cell in turn,

$$\frac{2G\theta}{l} = \frac{\tau_1 z_1 + \tau_3 z_3}{A_1}$$

$$= \frac{\tau_2 z_2 - \tau_3 z_3}{A_2}$$

where z_1, z_2 and z_3 are the mean perimeters ABC, CDA and CA respectively, the negative sign indicates a traverse against the direction of stress.

Example 14.1 For the cellular section shown in Fig. 14.3.

A_1 = 20 mm × 40 mm,

A_2 = 40 mm × 40 mm

t_1 = 2 mm, t_2 = 1.5 mm, t_3 = 2.5 mm

If the torque on the section is 300 N-m, determine the maximum shear stress and the angle of twist per unit length. G = 30 GN/m^2.

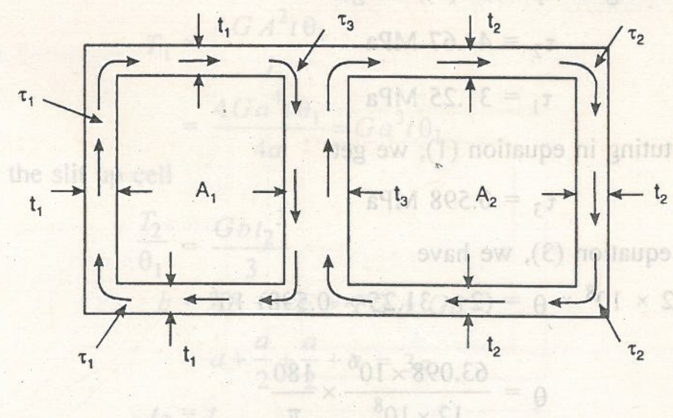


Fig. 14.3

Solution. $$2\tau_1 = 1.5\tau_2 + 2.5\tau_3 \qquad \text{...(1)}$$

$$300 = 2(\tau_1 \times 2 \times 10^{-3} \times 20 \times 40 \times 10^{-6} + \tau_2 \times 1.5 \times 10^{-3} \times 40 \times 40 \times 10^{-6})$$

$$3 = 32 \times 10^{-9}\tau_1 + 48 \times 10^{-9}\tau_2 \quad \ldots(2)$$

$$2 \times 30 \times 10^9 \times \theta = \frac{1}{20 \times 40 \times 10^{-6}}\left[\tau_1(40 + 2 \times 20)10^{-3} + \tau_3 \times 40 \times 10^{-3}\right]$$

$$= \frac{1}{20 \times 10^{-6}}(2\tau_1 + \tau_3)$$

$$\Rightarrow \quad 12 \times 10^8\theta = 2\tau_1 + \tau_3 \quad \ldots(3)$$

$$2 \times 30 \times 10^9 \times \theta = \frac{1}{40 \times 40 \times 10^{-6}}\left[\tau_2(40 \times 2 + 40)10^{-3} - \tau_3 \times 40 \times 10^{-3}\right]$$

$$= \frac{1}{40 \times 10^{-3}}(3\tau_2 - \tau_3)$$

$$\Rightarrow \quad 24 \times 10^8\theta = 3\tau_2 - \tau_3 \quad \ldots(4)$$

From Eqns. (3) and (4)

$$4\tau_1 = 3\tau_2 - 3\tau_3 \quad \ldots(5)$$

Solving equation (1) with (5) gives,

$$\tau_1 = \frac{3\tau_2}{4} \quad \ldots(6)$$

Substituting in equation (2), we get

$$\tau_2 = 41.67 \text{ MPa}$$

$$\tau_1 = 31.25 \text{ MPa}$$

Substituting in equation (1), we get

$$\tau_3 = 0.598 \text{ MPa}$$

From equation (3), we have

$$12 \times 10^8 \times \theta = (2 \times 31.25 \times 0.598)\, 10^6$$

$$\theta = \frac{63.098 \times 10^6}{12 \times 10^8} \times \frac{180}{\pi}$$

$$= 3.013 \text{ deg/m.}$$

Example 14.2 A thin walled section is shown in Fig. 14.4. It has a constant wall thickness t and one compartment is slit open. Find the stiffness of the section and value of maximum shear stress for a given torque.

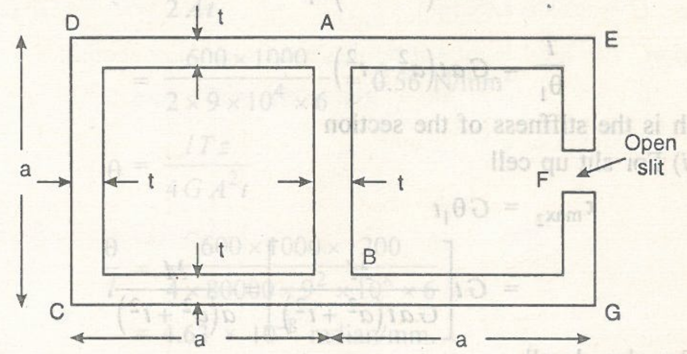


Fig. 14.4

Solution. (*a*) Let T be the total torque on the section for which θ is the twist per unit length. Let T_1 be the torque taken by the closed cell $ABCD$. Suppose the slit up box carries the torque T_2.

$$T = T_1 + T_2$$

For the closed cell portion

$$A = a^2 \text{ and } L = 4a$$

$$\tau = \frac{T_1}{2A_1 t_1}$$

and

$$\frac{T_1}{\theta_1} = \frac{4GA^2 t}{L}$$

where θ is the twist of the whole section per unit length.

$$T_1 = \frac{4GA^2 t\theta_1}{L}$$

$$= \frac{4Ga^4 t\theta_1}{4a} = Ga^3 t\theta_1$$

For the slit up cell

$$\frac{T_2}{\theta_1} = \frac{Gbt_2^{\,3}}{3}$$

$$b = AE + EF + FG + GB$$

$$= a + \frac{a}{2} + \frac{a}{2} + a = 3a$$

and

$$t_2 = t$$

$$T_2 = \frac{G(3a)t^3\theta_1}{3} = Gat^3\theta_1$$

$$T = T_1 + T_2$$
$$= Ga^3 t\theta_1 + Gat^3\theta_1$$
$$= Gat\left(a^2 + t^2\right)\theta_1$$

$$\Rightarrow \quad \frac{T}{\theta_1} = Gat\left(a^2 + t^2\right)$$

which is the stiffness of the section

(*b*) (*i*) For slit up cell

$$\tau_{max_2} = G\theta_1 t$$
$$= Gt\left[\frac{M}{Gat\left(a^2+t^2\right)}\right] = \frac{M}{a\left(a^2+t^2\right)}$$

(*ii*) For closed cell

$$\tau_{max_1} = \frac{2G\theta_1 A_1}{L_1} = \frac{2G\theta_1 a^2}{4a}$$
$$= \frac{Ga}{2}\left[\frac{T}{Gat\left(a^2+t^2\right)}\right]$$

$$\Rightarrow \quad \tau_{max_1} = \frac{T}{2t\left(a^2+t^2\right)}$$

$$\tau_{max} = \tau_{max_1} = \frac{T}{2t\left(a^2+t^2\right)}$$

Example 14.3 A closed cellular square section is subjected to a torque 600 Nm. Find the maximum shear stress and the twist per unit length, neglect stress concentration. $G = 8 \times 10^4$ MPa

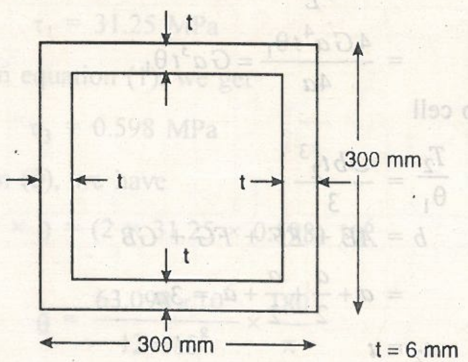


Fig. 14.5

Solution. (*a*) $t = 6$ mm

$$A = 300 \times 300 = 9 \times 10^4 \text{ mm}^2$$
$$\tau = \frac{T}{2At}$$
$$= \frac{600 \times 1000}{2 \times 9 \times 10^4 \times 6} = 0.56 \text{ N/mm}^2$$

(*b*)
$$\theta = \frac{lTz}{4GA^2t}$$

$$\Rightarrow \quad \frac{\theta}{l} = \frac{600 \times 1000 \times 1200}{4 \times 80000 \times 9^2 \times 10^8 \times 6}$$
$$= 4.63 \times 10^{-8} \text{ radian/mm.}$$

Example 14.4 A thin walled member 1 m long has the cross section shown in Fig. 14.6. Determine the maximum torque which can be carried by a section if the angle of twist is limited to 10°. What will be the maximum shear stress when this maximum torque is applied ? G = 80 GPa.

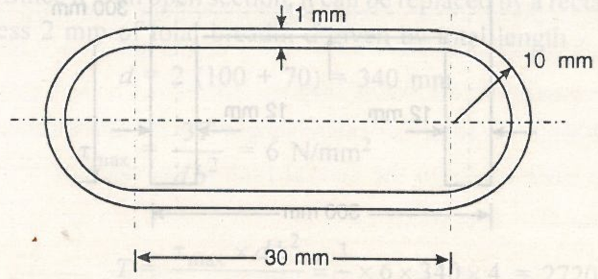


Fig. 14.6

Solution. Perimeter $z = (2 \times 30 + 2\pi \times 10) = 122.83$ mm

Area enclosed $A = (20 \times 30 + \pi \times 10^2) = 914.16 \text{ mm}^2$

$$\theta = \frac{Tlz}{4A^2Gt}$$

$$\frac{10 \times \pi}{180} = \frac{T \times 1 \times 122.83 \times 10^{-3}}{4(914.16)^2 10^{-12} \times 80 \times 10^9 \times 1 \times 10^{-3}}$$

$$\Rightarrow \quad T = 380 \text{ N m}$$

$$\tau_{max} = \frac{T}{2At} = \frac{380}{2 \times 914.16 \times 10^{-6} \times 1 \times 10^{-3}}$$
$$= 207.84 \text{ MPa}$$

Example. 14.5 A 300 × 300 mm *I* section with flanges and web 12 mm thick is subjected to a torque of 615 N m. Find the maximum shear stress and the twist per unit length, neglecting the stress concentration.

(Take : G = 80000 MN/m^2) (UPTU 2001-02)

Solution. The total length of equivalent rectangular section

$$d = 300 + 300 + 276 = 876 \text{ mm}$$

Thickness b = 12 mm

$$\tau_{max} = \frac{3T}{db^2}$$

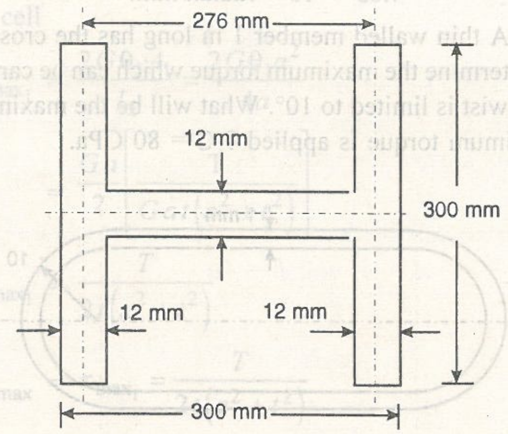


Fig. 14.7

$$= \frac{3(615 \times 1000)}{876 \times (12)^2} = 14.63 \text{ N/mm}^2$$

$$\frac{\theta}{l} = \frac{3T}{db^3G} \qquad \frac{\theta}{l} = \frac{3T}{db^2}\frac{1}{Gb}$$

$$\Rightarrow \quad \frac{\theta}{l} = \tau_{max}\left(\frac{1}{Gb}\right)$$

$$\Rightarrow \quad \frac{\theta}{l} = 14.63 \times \frac{1}{80000 \times 12} \text{ rad/mm}$$

$$= 0.015 \text{ rad/m.}$$

Example 14.6 An open rectangular section is acted upon by twisting moment such that the shear stress induced in it is 6 MPa. Find the value of the twisting moment.

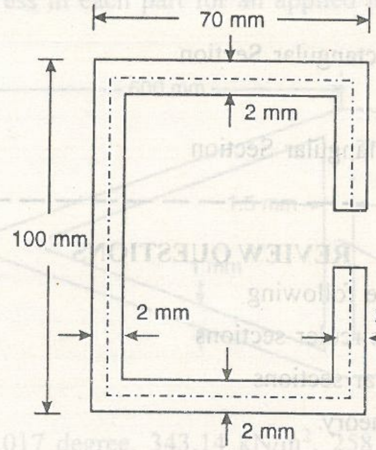


Fig. 14.8

Solution. Since it is an open section, it can be replaced by a rectangular section of thickness 2 mm of total breadth d given by total length

$$d = 2\,(100 + 70) = 340 \text{ mm}$$

$$\tau_{max} = \frac{3T}{db^2} = 6 \text{ N/mm}^2$$

$$\Rightarrow \quad T = \frac{\tau_{max} \times db^2}{3} = \frac{1}{3} \times 6 \times 340 \times 4 = 2720 \text{ N-mm}$$

IMPORTANT DERIVATIONS

1. What do you understand by Bredt-Batho Theory ? Consider a cellular section under torsion shown in Fig. 14.2. Find the angle of twist per unit length.

   Hint : Show that, $\frac{\theta}{l} = \frac{\tau_1 z_1 + \tau_2 z_2}{2A_1 G} = \frac{\tau_2 z_2 - \tau_3 z_3}{2A_2 G}$

2. A closed tube of small thickness shown in Fig. 14.1 is acted upon by a torque T in a transverse plane. Show that angle of twist per unit length will be given by

$$\frac{\theta}{l} = \frac{Tz}{4GA^2t}\,.$$

USEFUL RESULTS

1. $\tau = \dfrac{T}{2At}$ 2. $\theta = \dfrac{Tlz}{4GA^2t}$

3. $\tau_{max} = \dfrac{3T}{db^2}$ Rectangular Section

4. $\dfrac{\theta}{l} = \dfrac{3T}{db^3G}$ Rectangular Section

REVIEW QUESTIONS

Write short notes on the following

(*i*) Torsion of non circular sections

(*ii*) Torsion of cellular sections

(*iii*) Saint Venant's theory.

NUMERICAL PROBLEMS

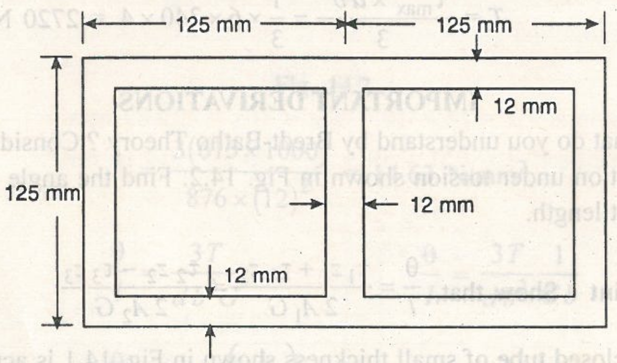
1. A built-up steel plate girder has the section shown in Fig. The thickness of the plate is 12 mm all round. If the maximum allowable shear stress in the material is 65 MPa. Find

(*i*) the maximum allowable torque,

(*ii*) the angle of twist per metre length due to this torque,

(*iii*) the shear stress in the central limb of the section.

Take G = 8 × 10⁴ MPa.

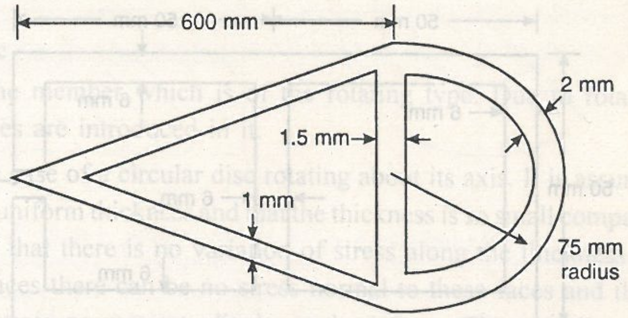


[**Ans.** (*i*) 4062.5 Nm (*ii*) 8.12 × 10⁻⁴ rad/m (*iii*) zero]

2. A 30 cm *I* beam with flanges and with a web 1.25 cm thick, is subjected to a torque *T* = 4900 Nm. Find the maximum shear stress and the angle of twist per unit length.

[**Ans.** 63602 kPa = 1503/*G* radians per cm length]

3. The cross section of an aeroplane elevator is shown in Fig. If the elevator is 2 m long and constructed from aluminium alloy with *G* = 30 GPa, Calculate the total angle of twist of the section and the magnitude of the shear stress in each part for an applied torque of 40 Nm.



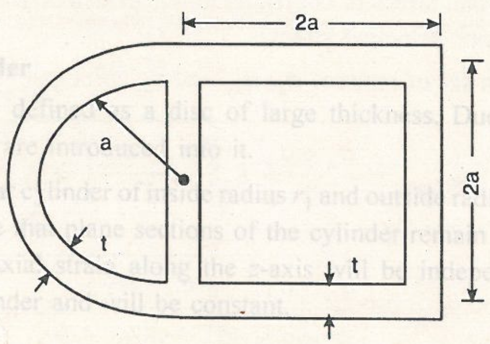
[**Ans.** .017 degree, 343.14 kN/m², 258 kN/m², 115.24 kN/m²]

4. A 40 mm × 20 mm rectangular steel shaft is subjected to a torque of 1 kNm. What is the magnitude of the maximum shear stress set up in the shaft and the corresponding angle of twist per unit length of the shaft ? *G* = 80 GN/m². [**Ans.** 254 MPa, 9.77 deg/m]

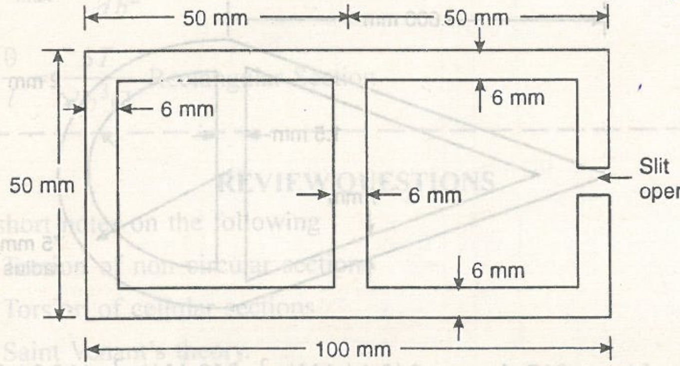
5. A thin walled box shown in figure is subjected to a torque *T*. Determine the shear stresses in the walls and the angle of twist per unit length of the box.

Ans. $q_1 = \dfrac{(\pi+2)T}{a^2\left(\pi^2+12\pi+16\right)}, \quad q_2 = \dfrac{5\pi+8}{5\pi+18}q_1,$

$$\theta = \frac{(2\pi+3)T}{2Ga^3t\left(\pi^2+12\pi+16\right)}$$



6. A mild steel built up section, shown in figure is acted upon by a twisting moment so as to induce in it the maximum shear stress of 67 MPa. Find the value of the twisting moment and the shear stress in various parts of the section. Take $G = 80000$ MPa.



[Ans. 10074 Nm, 4.02 MPa, 33.5 MPa, 67 MPa]

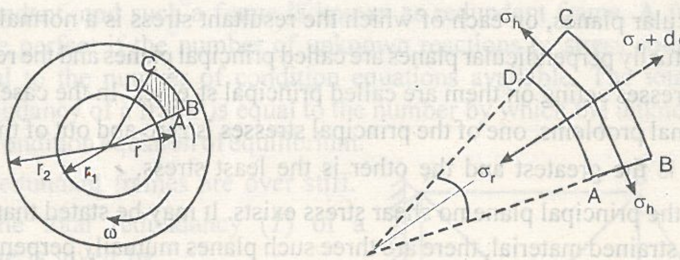
Model Short Notes

1. Rotating Disc

Disc is a machine member which is of the rotating type. Due to rotation centrifugal stresses are introduced in it.

Let us take the case of a circular disc rotating about its axis. It is assumed that the disc is of uniform thickness and that the thickness is so small compared with its diameter that there is no variation of stress along the thickness. At the free flat surfaces there can be no stress normal to these faces and there can be no shear stress on or perpendicular to these faces. Thus the direction of axis is the direction of zero principal stress. The radial and circumferential stresses represent the principal stresses.

Let us consider a flat rotating disc of uniform thickness t. Let r_1 and r_2 be the inner and outer radii of the disc. The disc is rotating at ω speed.

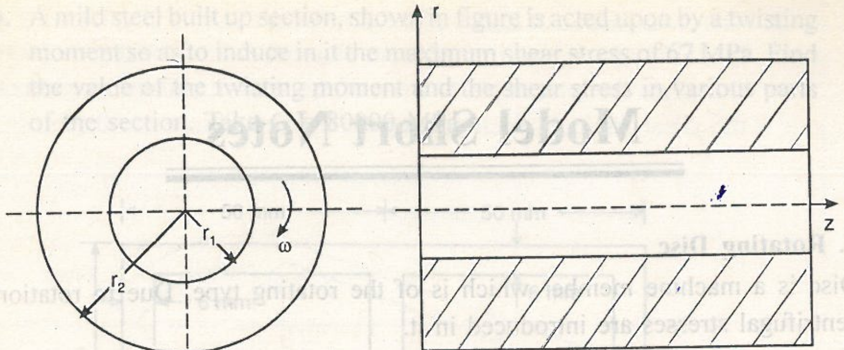


An element of the disc $ABCD$ at radius r is acted upon by stresses σ_r and $\sigma_r + d\sigma_r$ on faces AD and BC respectively and by stresses σ_h on the faces AB and CD.

2. Rotating Cylinder

A cylinder may be defined as a disc of large thickness. Due to rotation, centrifugal stresses are introduced into it.

Consider a circular cylinder of inside radius r_1 and outside radius r_2 rotating at speed ω. Assume that plane sections of the cylinder remain plane during rotation, then the axial strain along the z-axis will be independent of the radius r of the cylinder and will be constant.



$$\text{Radial strain } \varepsilon_r = \frac{1}{E}[\sigma_r - \nu(\sigma_h - \sigma_z)]$$

$$\text{Hoop strain } \varepsilon_h = \frac{1}{E}[\sigma_h - \nu(\sigma_r + \sigma_z)]$$

$$\text{Axial strain } \varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_r + \sigma_h)]$$

3. Principal Stresses and Principal Planes

At any point within a stressed body, there always exist three mutually perpendicular planes, on each of which the resultant stress is a normal stress. These mutually perpendicular planes are called principal planes and the resultant normal stresses acting on them are called principal stresses. In the case of two dimensional problems, one of the principal stresses is zero and out of the other two, one is the greatest and the other is the least stress.

Along the principal plane no shear stress exists. It may be stated that at any point in a strained material, there are three such planes mutually perpendicular to each other carrying direct stresses only and no tangential stress. Out of the three, the plane carrying the maximum normal stress is called the major principal plane and the stress is called the major principal stress. The plane carrying the minimum normal stress is known as minor principal plane and the stress is known as minor principal stress.

For a given set of stresses σ_x, σ_{xy} and τ_{xy} principal stresses are given by

$$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

$$\sigma_{min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

4. Castigliano's Theorem

This theorem is extremely useful for finding displacements of elastic bodies subjected to axial loads, torsion, bending or any combination of these loadings. The theorem states that the partial derivative of the total internal strain energy with respect to any externally applied force yields the displacement under the point of application of that force in the direction of that force. The terms force and displacement are used in generalised sense and could either indicate a usual force and its linear displacement or a couple and the corresponding angular displacement.

This theorem says that

$$\delta_n = \frac{\partial U}{\partial P_n}$$

Castigliano's theorem is extremely useful for determining the indeterminate reactions. This theorem can be applied to each reaction and we can know the displacement corresponding to each reaction before hand. In this manner it is possible to establish as many equations as there are redundant reactions.

After the values of all reactions are found, the deflection at any desired point can be found by direct use of Castigliano's theorem.

5. Redundant Frames

The excess member or restraints in a frame or structure are described as redundant, and such a frame is known as redundant frame. A frame is said to be perfect if the number of unknown reactions or stress components are equal to the number of condition equations available. The total degree of redundancy of a frame is equal to the number by which the unknowns exceed the condition equation of equilibrium.

Redundant frames are over stiff.

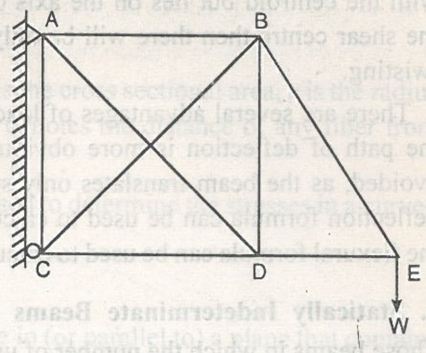
The total redundancy (T) of a frame is given by

$$T = m - (2j - R)$$

where m = total number of members

j = total number of joints

R = total number of reaction components



The frame in figure has

$$R = (2 + 1) = 3$$

$$m = 8, j = 5$$

$$\therefore \quad T = 8 - (2 \times 5 - 3)$$

$$= 1$$

Thus the frame is redundant to single degree.

6. Winkler-Bach Theory

This theory is used to determine the stresses in a curved beam. The following assumptions are made in this analysis :

1. Plane transverse sections before bending remain plane after bending.
2. Limit of proportionality is not exceeded.
3. Radial strain is negligible.
4. The material considered is isotropic and obeys Hooke's law.

The bending stresses in a curved beam are given by the following equations

$$\sigma = \frac{M}{AR}\left[1+\frac{R^2}{h^2}\left(\frac{y}{R+y}\right)\right] \quad \text{tensile}$$

$$\sigma = \frac{M}{AR}\left[1-\frac{R^2}{h^2}\left(\frac{y}{R-y}\right)\right] \quad \text{compressive}$$

where R is radius of curvature of the centroidal axis and y is distance of fiber from the centroidal axis.

7. Shear Centre

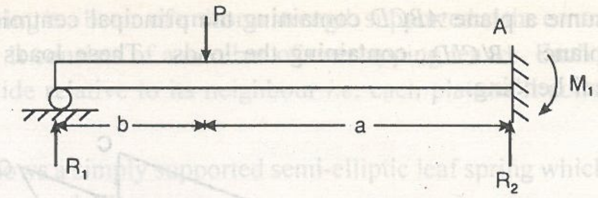
Shear centre is a point in the cross section of every elastic beam, through which transverse forces may be applied so as to produce bending only, with no torsion of the beam. It is the point of intersection of the bending axis and the plane of the transverse section. Shear centre is also called the centre of twist. If a beam has two axes of symmetry, then shear centre coincides with the centroid. For sections having one axis of symmetry, the shear centre does not coincide with the centroid but lies on the axis of symmetry. If a load passes through the shear centre then there will be only bending in the cross section and no twisting.

There are several advantages of loading a beam at the shear centre. First, the path of deflection is more obvious so that clearance problems can be avoided, as the beam translates only straight forward. Second, the standard deflection formula can be used to calculate the amount of deflection. Third, the flexural formula can be used to calculate the stresses and stain in the beam.

8. Statically Indeterminate Beams

Those beams in which the number of unknown reactions exceeds the number of equilibrium equations available, are said to be statically indeterminate. In such a case it is necessary to supplement the equilibrium equations with additional equations evolved from the deformations of the beam.

In the case of a beam fixed at one end and supported at the other, we have unknown reactions R_1, R_2 and M_1. The two statics equations must be supplemented by one equation based upon deformation.



The two statics equations are :

$$\Sigma M_A = M_1 - Pa + R_1(a+b)$$

$$\Sigma F_y = R_1 + R_2 - P = 0$$

These are two equations in the three unknowns R_1, R_2 and M_1.

Hence this is a statically indeterminate beam.

9. Curved Beam

Machine members and structures such as hooks, links and rings etc. which have large initial curvature are known as curved beams. Occasionally initially curved beams are encountered in machine design and other areas.

Unlike the initially straight beams, the simple bending formula is not applicable for the curved beams as their neutral axis does not coincide with the centroidal axis. Their neutral axis shifts towards the centre of curvature of the beam by a distance $\bar{y}$. The bending stress for a curved beam is given by

$$\sigma = \frac{My}{A\bar{y}(r+y)}$$

where M is the bending moment, A is the cross sectional area, r is the radius of curvature of the neutral axis and y denotes the distance of any fiber from the neutral axis.

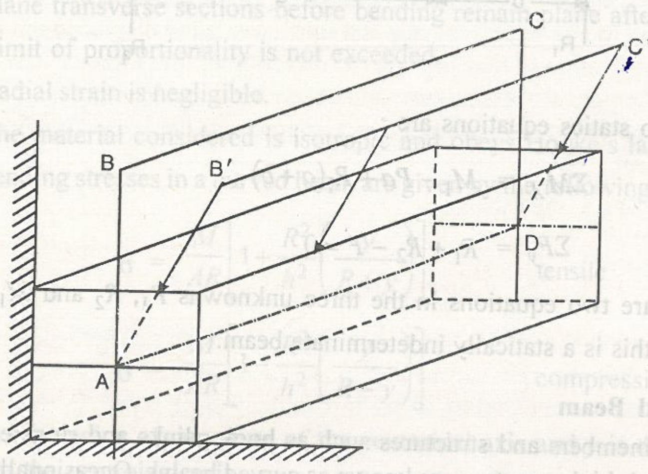
Generally, Winkler Bach theory is used to determine the stresses in a curved beam.

10. Unsymmetrical Bending

Bending caused by loads that do not lie in (or parallel to) a plane that contains the principal centroidal axes of inertia of the cross section is called unsymmetrical bending.

Frequently beams are of unsymmetric cross section or even if the cross-section is symmetric the plane of the applied load may not be one of the planes of symmetry. In either of these cases the expression $\sigma = My/I$ is not valid for determination of the bending stress.

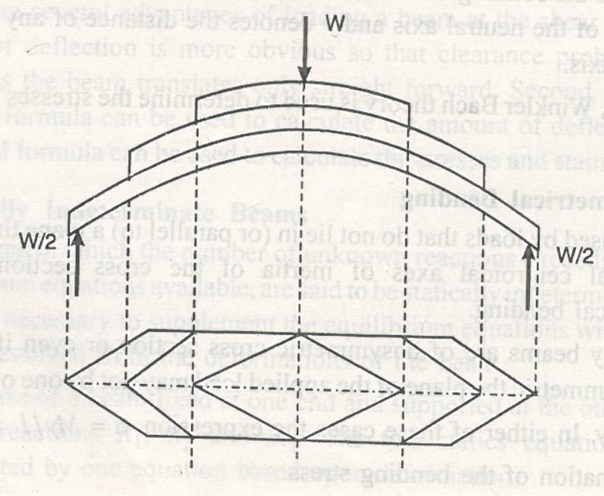
Let us assume a plane $ABCD$ containing the principal centroidal axes of inertia and plane $AB'C'D$, containing the loads. These loads will cause unsymmetrical bending.



Example of unsymmetrical bending are angle sections, *I*-section and channel sections which are used as perlins in trusses. To determine the deflection of a beam due to unsymmetrical bending the bending moment may be resolved into components parallel to the principal planes.

11. Leaf Spring

A leaf spring consists of number of parallel strips of metal of same width, placed one above the other. The plates are bent to the same radius so that they contact only at their edges. When the load W is applied at the centre, the change of curvature of each plate is uniform and the same for all the plates and the contact will continue to be at the ends only.



A leaf spring is a beam of uniform strength supported at the centre and loaded at the ends. It consists of a number of overlapping leaves. Each plate or leaf is free to slide relative to its neighbour *i.e.* each plate will act as a separate beam.

Figure shows a simply supported semi-elliptic leaf spring which is centrally loaded with a load W.

This type of spring is commonly used in carriages such as railway wagons, cars etc.

12. Helical Spring

A helical spring is a piece of wire coiled in the form of helix. If the slope of the helix of the coil is so small that the bending effects can be neglected, then the spring is called a close coiled helical spring. In such a spring only torsional shear stresses are introduced. On the other hand, if the slope of the helix of the coil is quite appreciable, then both the bending as well as torsional shear stresses are introduced in the spring and a spring of this type is called an open coiled helical spring.

Close coiled helical spring is so closely coiled that each turn is practically a plane at right angle to the axis of the helix and the stresses upon the material are almost of pure torsion.

In the case of open coiled helical spring, the coils are not close together. The bending couple can not be considered negligible in compression with the torsion couple.

13. Airy's Stress Function

Let
$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2}$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

Then these equations satisfy the equilibrium equations identically in the absence of the body forces. ϕ is known as the Airy's stress function.

The solution of a two dimensional problem of elasticity reduces to the integration of the differential equations of equilibrium together with the compatibility equation and the above three equations. It can be seen that Airy's stress function ϕ satisfies the equilibrium equations.

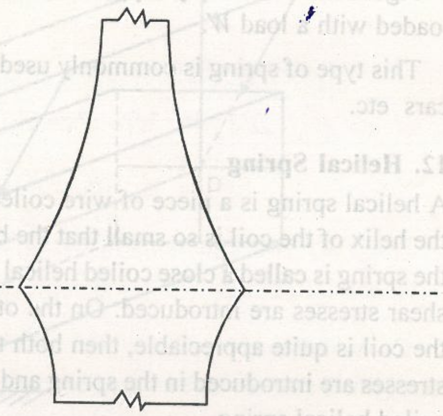
14. Disc of Uniform Strength

A disc of uniform strength is the one in which the values of radial and hoop stresses are equal in magnitude for all values of r.

Hence $\sigma_h = \sigma_r = \sigma =$ constant

This suggests that the disc of uniform strength must have a varying thickness as shown in figure. If t be the thickness of such disc at radius r, $t = t_0$ at $r = r_1$

then $t = t_0 e^{-\rho\frac{\omega^2}{2\sigma}(r^2 - r_1^2)}$ gives



the thickness of disc at any radius.

15. Compatibility Equations

There are six independent stress components acting at a point and the complete solution of the problem requires the determination of these six stress components. Thus there are six unknowns and only three equations of equilibrium available. These equations of static equilibrium must be supplemented with equations of compatibility of deformations to get the complete solution. These equations are given by

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left[-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right]$$

$$2\frac{\partial^2 \varepsilon_y}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right]$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y}\right]$$

16. Principle of Three Moments

If the moments over the intermediate supports of a continuous beams are known then the bending moment diagram can be drawn easily. The moments over the intermediate supports are determined by using principle of three moments which is also known as Clapeyron's theorem of three moments. It states that if BC and CD are any two consecutive span of a continuous beam subjected to an external loading, than the moments M_B, M_C and M_D at the supports B, C and D are given by

$$M_B L_1 + 2M_C(L_1 + L_2) + M_D L_2 = \frac{6a_1\bar{x}_1}{L_1} + \frac{6a_2\bar{x}_2}{L_2}$$

where L_1 = Length of span BC

L_2 = Length of span CD

a_1 = area of B.M. diagram due to vertical loads on span BC

a_2 = area of B.M. diagram due to vertical loads on span CD.

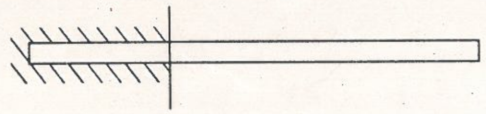
$\bar{x}_1$ = distance of C.G. of the bending moment diagram due to vertical loads on BC from B.

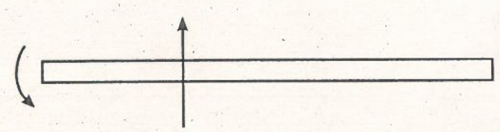
$\bar{x}_2$ = Distance of C.G. of the B.M. diagram due to vertical load on CD from D.

17. Saint Venant's Principle

This principle states that the stresses and strains at a point sufficiently away from the applied load are not significantly changed if the load is replaced by another statically equivalent load. We can also say that if the forces acting on a small portion of the surface of an elastic body are repalced by a statically equivalent load, the stresses developed may vary locally, but the stresses at a distance sufficiently away from this area remain almost unchanged.

For example, the complex supporting force system exerted by the wall on the cantilever beam can be replaced by a single force and a couple, to simplify the computation of stresses and strains on the region towards the right end of cantilever.





In other words, it can be said that the manner of force applicaiton on stresses is important only in the vicinity of the region where the force is applied, elsewhere the average stress can be assumed to be constant.