ADVANCED STRENGTH OF MATERIALS

20 Model Short Notes
110 Solved Problems

ALOK GUPTA

UMESH PUBLICATIONS

Foreword

Since the advent of modern hi-tech material and machinery system, the patterns and process of study have been rapidly changing. There is an emergence of new subjects to meet the ensuing demands arising day by day in the so called modern complex and composite structure of machine technology.

Advanced Strength of Materials has emerged as one of the most vital subjects of study for the students of Mechanical, Industrial, Metallurgy and Industrial Production Engineering in recent years. It is obviously a result of advancement of knowledge in Mechanical Engineering related areas. Since this subject has been added in the university curriculum, there is a need of good books on the subject for students as well as for teachers.

The book "Advanced Strength of Materials" written by Shri Alok Gupta has been designed keeping in mind the requirements of the learners. This book is innovative, exhaustive and unique in nature. It is endowed with substantial matter on the subject. It opens new vistas in the mind of learners. The author has carefully avoided undue material as well as over lapping and made the book very useful for students and teachers who are dealing with the subjects.

The author has been intelligent enough in putting the matter in precise way without disturbing the authenticity of the subject matter. The entire subject matter has been discussed in 14 chapters. The intricacies of the subject have been presented in a very lucid manner.

I hope it would be proved an excellent endeavour of the author to write a book on this advanced subject. This book will certainly fulfil the aspiration of the students and teachers and undoubtedly the students will find it easy to study the subject through this book.

(Dr. S. N. Chauhan)
Director
S. D. College of Engg. & Tech.
Muzaffarnagar

Contents

1. Elasticity Theory 1
1.1 Introduction 1
1.2 Three Dimensional Stresses 1
1.2.1 Normal and Shear Stresses 3
1.2.2 Principal Stresses 4
1.2.3 Principal Directions 5
1.3 Strain Tensor : 3D Strain 5
1.3.1 Normal and Shearing Strains 7
1.3.2 Principal Strains 8
1.3.3 Principal Strain Directions 8
1.4 Relation in between Three Dimensional Stress and Strain (Generalised Hooke's Law) 9
1.5 Equilibrium Equations 11
1.6 Compatibility Equations 12
1.6.1 Compatibility Equation in Two Dimensional Case 13
1.7 Airy's Stress Function 16
Useful Results 26
Review Questions 27
Numerical Problems 27
2. Castigliano's Theorem 28
2.1 Introduction 28
2.2 Statement of Castigliano's Theorem 28
2.3 Castigliano's Theorem 28
2.3.1 Deflection under Axial Load 30
2.3.2 Deflection under Bending 31
2.3.3 Rotation under Torsion 31
2.3.4 Rotation under Bending 31
2.3.5 Deflection under Torsion 31
2.3.6 Deflection under Shear 31
Summary 37
Review Questions 38
Numerical Problems 38

3. Statically Indeterminate Structures 39
3.1 Introduction 39
3.2 Types of Statically Indeterminate Structures 44
3.3 Stresses in Simple Statically Indetermine Structures 44
3.4 Stresses in Indeterminate Structures Supporting a Load 46
3.5 Stresses in Composite Structures of Equal Lengths 49
3.6 Stresses in Composite Structures of Unequal Lengths 51
3.7 Statically Indeteminate Beams 53
3.8 A Short Note on Condition for Statistical Indeterminancy 54
Summary 55
Review Questions 57
Numerical Problems 57

4. Use of Energy Methods for Solving Indeterminate Beam Problems 59
4.1 Introduction 59
4.2 Review of Castigliano's Theorem 59
4.3 Application of Castigliano's Theorem to Statically Indeterminate Problems 62
Review Questions 67
Numerical Problems 68

5. Deflection of Beams by Integration Method 71
5.1 Introduction 71
5.2 Differential Equation for Deflection 71
5.3 Other Useful Equations 73
5.4 Integration Method 74
5.5 A few General Cases 74
5.5.1 Cantilever Subjected to Moment at Free End 74
5.5.2 Cantilever Subjected to Concentrated Load at Free End 76
5.5.3 A Cantilever Subjected to Uniformly Distributed Load 77
5.5.4 A cantilever Subjected to have Varying Linearly from Zero at Free End to w/unit Length at Fixed End 78
5.5.5 Simply Supported Beam Subjected to a Central Concentrated Load 79
5.5.6 Simply Supported Beam Subjected to Uniformly Distributed Load 81
5.5.7 A Simply Supported Beam Subjected to a Load Varying Linearly from Zero at One End to *w*/unit Length at Other End 82
Useful Results 91
Review Questions 92
Numerical Problems 92

6. Principle of Three Moments 94
6.1 Introduction 94
6.2 Claypeyron's Theorem of Three Moments 94
6.3 Application of Claypeyron's Equation of Three Moments to Continuous Beam with Simply Supported Ends 98
6.4 Application of Claypeyron's Equation of Three Moment to Continuous Beams with Fixed Ends 106
6.5 Application of Claypeyron's Equation of Three Moments to Continuous Beam with Orverhanging End Span 111
Summary 115
Review Questions 116
Numerical Problems 116

7. Redundant Frames 118
7.1 Introduction 118
7.2 Redundant Frames 120
7.3 Degree of Redundancy 120
7.4 Analysis of Redundant Frames 122
Useful Results 127
Review Questions 127
Numerical Problems 127

8. Bending of Beams with Large Initial Curvature 129
8.1 Introduction 129
8.2 The Winkler-Bach Theory 130
8.3 Position of Neutral Axis 135
8.4 Values of h^2 135
8.5 Beams with Rectangular Cross-section 136
8.6 Beams of Trapezoidal Cross-section 137
Important Results 142
Review Questions 142
Numerical Problems 143

9. Stresses in Crane Hook, Circular Rings and Chain Links 145
9.1 Introduction 145
9.2 Crane Hook 145
9.3 Stresses in a Ring 146
9.4 Stresses in a Chain Link 150
Expected Derivations 160
Numerical Problems 160

10. Unsymmetrical Bending 163
10.1 Introduction 163
10.2 Unsymmetrical Bending 163
10.3 Properties of Beam Cross-section 164
10.4 Stresses due to Unsymmetrical Bending 166
10.5 Location of Neutral Axis 167
10.6 Deflection of Straight Beams due to Unsymmetrical Bending 168
10.7 Short Notes on Neutral Axis 169
Useful Results 176
Review Questions 177
Numerical Problems 177

11. Shear Centre 179
11.1 Introduction 179
11.2 Flexural axis or Bending Axis 181
11.3 Shear Centre for Channel Section 181
11.4 Shear Centre for I-section 183
Useful Results 188
Expected Derivations/Explanations 188
Review Quesitons 188
Numerical Problems 189

12. Rotating Discs and Cylinders 190
12.1 Introduction 190
12.2 Rotating Disc 190
12.2.1 Strain Considerations 191
12.3 Hollow Disc (Disc with a central hole) 193
12.4 Solid Disc 194
12.5 Disc of Uniform Strength 195
12.6 Rotating Cylinder 197
12.7 Solid Cylinder 200
12.8 Hollow Cylinder 201
Expected Derivations 207
Useful Results 208
Numerical Problems 209

13. Springs 210
13.1 Introduction 210
13.2 Helical Spring 210
13.3 Closed Coil Helical Spring 210

13.4 Closed Coil Helical Spring under Axial Load 211
13.5 Helical Springs of Non Circular Wire 213
13.6 Closed Coil Helical Spring under Axial Torque 214
13.7 Strain Energy in the Spring 215
13.8 Springs in Series 216
13.9 Springs in Parallel 216
13.10 Springs under Impact Load 217
13.11 Open Coiled Helical Spring under Axial Load 217
13.12 Open Coiled Helical Spring under Axial Twist 219
13.13 Stresses in the Spring Wire 221
13.14 Axial Load and Twisting Moment acting Simultaneously on Circular Cross section 223
13.15 Deflection of Spring by Energy Method 224
13.16 Leaf Spring 225
Expected Derivations 241
Useful Results 242
Review Questions 243
Numerical Problmes 244

14. Torsion of Non Circular and Hollow Sections 245
14.1 Introduction 245
14.2 Rectangular Sections 245
14.3 Equilateral Triangular Section 246
14.4 Elliptical Sections 246
14.5 Saint-Venant's Theory of Twisting of Non-circular Shafts 247
14.6 Torsion of Thin Tubular Section 247
14.7 Torsion of Thin-Walled Cellular Sections 248
Important Derivations 255
Useful Results 256
Review Questions 256
Numerical Problems 256

Model Short Notes 259
1. Rotating Disc 259
2. Rotating Cylinder 259
3. Principal Stresses and Principal Planes 260
4. Castigliano's Theorem 261
5. Redundant Frames 261
6. Winkler-Bach Theory 262

7. Shear Centre 262
8. Statically Indeterminate Beams 262
9. Curved Beam 263
10. Unsymmetrical Bending 263
11. Leaf Spring 264
12. Helical Spring 265
13. Airy's Stress Function 265
14. Disc of Uniform Strength 266
15. Compatibility Equations 266
16. Principle of Three Moments 267
17. Saint Venant's Principle 267

1

Elasticity Theory

1.1 INTRODUCTION

In your third semester course of 'Strength of Materials' (ME-303) you have covered two dimensional stress system without any reference to the co-ordinate system. However, in actual practice, in quite a number of large cases the stresses do not lie in one plane and the the third principal stress is not zero. Such a state of stress where the stresses along the three principal axes exist, is known as the three dimensional or tri-axial state of stress.

In this chapter we will not only discuss the concepts of 3-dimensional stress and strain, but also cover their relationship expressed by 'Generalised Hooke's Law'. While studying about 3-dimensional stress-system, we will find that there are 6 independent stress components acting at a point and the complete solution of the problem requires the determination of these six stress components. Thus, there are 6 unknowns and only three equations of equilibrium will be available. Hence, to get the solution 'Compatibility Equations' will be introduced.

In this chapter we will see that these three 'Compatibility Equations' may be reduced to one single equation in terms of 'Stress Function ϕ' which is popularly known as 'Airy's Stress Function'.

1.2 THREE DIMENSIONAL STRESSES

On an elastic body, two type of forces act : body forces and surface forces. Dimensionally surface force is defined as force per unit area, while forces distributed over the volume of a body, such as magnetic forces, gravitational forces etc. are taken as force per unit volume. These are called body forces.

Thus, there rises a need to express the total stress field on any three dimensional element using a square matrix. This square matrix of stresses is known as '*Stress tensor*'. It may be written as

$$\tau_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{y} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{z} \end{bmatrix}$$

We see that in the case of three dimensions, there are nine stresses, three normal and six shear. These stress components have been shown on a rectangular parallelopiped in Fig. 1.1.

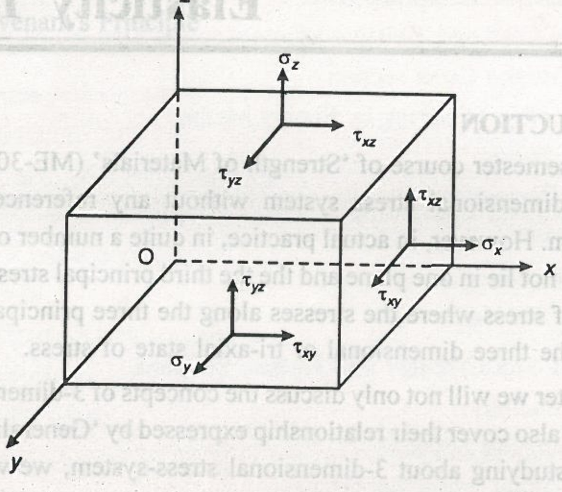


Fig. 1.1

The first subscript indicates the direction of the outward drawn normal and the second subscript indicates the direction of the co-ordinate axis. For example σ_{yy} (or simply σ_y) signifies the normal stress acting on the face of the element that is perpendicular to y-axis and the stress is acting in the y-direction. The shear stress τ_{xz} denotes a stress acting on the face of an element that is perpendicular to x-axis, the stress acting in the direction of z-axis.

Considering the equilibrium of an elemental volume of size dx, dy and dz and applying $\Sigma M = 0$ about each of x, y and z axis, we can see that out of six shearing stresses there are only three independent shearing stresses. Taking $\Sigma M_z = 0$ (where M_z is the moment about z-axis).

$$(\tau_{yx}.dx.dz)dy = (\tau_{xy}.dy.dz)dx$$

or $$\tau_{yx} = \tau_{xy}$$

In the same way it can be shown that

$$\tau_{yz} = \tau_{zy}$$

and $$\tau_{zx} = \tau_{xz}$$

Thus, we see that on account of the complementary nature of the shear stresses, we are left with *only six independent stress components i.e.* σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} and τ_{zx}.

1.2.1 Normal and Shear Stresses

Let us consider a rectangular parallelopipe (Fig. 1.2) subjected to the three-dimensional stress system. Let σ_r be the resultant stress on a plane passing through the point O and σ_{rx}, σ_{ry} and σ_{rz} be its components along the three axes of reference.

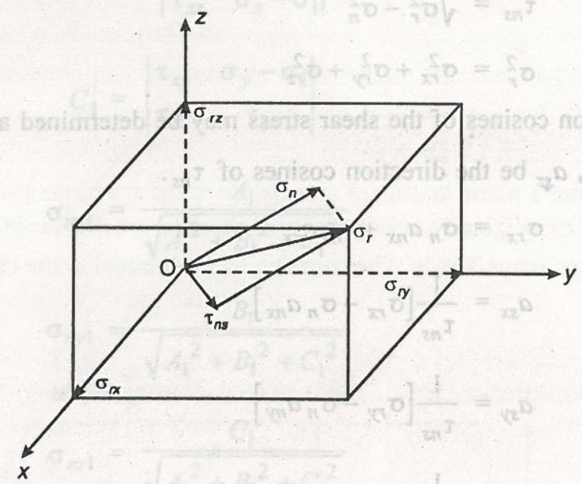


Fig. 1.2

The direction cosines of the resultant stress are

$$a_{rx} = \frac{\sigma_{rx}}{\sigma_r}, \quad a_{ry} = \frac{\sigma_{ry}}{\sigma_r}, \quad a_{rz} = \frac{\sigma_{rz}}{\sigma_r}$$

Let σ_n and τ_{ns} be the normal and shear stresses in the plane whose normal $\bar{n}$ has direction cosines a_{nx}, a_{ny}, a_{nz}.

Now $$\sigma_n = \sigma_r . a_{nr}$$

where $$a_{nr} = a_{nx}.a_{rx} + a_{ny}.a_{ry} + a_{nz}.a_{rz}$$

$$\therefore \quad \sigma_n = \sigma_r(a_{nx}a_{rx} + a_{ny}a_{ry} + a_{nz}a_{rz})$$

$$= \sigma_{rx}.a_{nx} + \sigma_{ry}.a_{ny} + \sigma_{rz}.a_{nz}$$

Resolving the forces acting on the parallelopipe along the co-ordinate axes:

$$\sigma_{rx} = \sigma_x a_{nx} + \tau_{xy} a_{ny} + \tau_{xz} a_{nz}$$

$$\sigma_{ry} = \tau_{xy} a_{nx} + \sigma_y a_{ny} + \tau_{yz} a_{nz}$$

$$\sigma_{rz} = \tau_{xz} a_{nx} + \tau_{yz} a_{ny} + \sigma_z a_{nz}$$

$$\therefore \quad \sigma_n = \sigma_x a_{nx}^2 + \sigma_y a_{ny}^2 + \sigma_z a_{nz}^2 + 2(\tau_{xy} a_{nx} a_{ny} + \tau_{yz} a_{ny} a_{nz} + \tau_{xz} a_{nx} a_{nz}) \quad ...(1)$$

Now $\sigma_r^2 = \sigma_n^2 + \tau_{ns}^2$

$$\therefore \quad \tau_{ns} = \sqrt{\sigma_r^2 - \sigma_n^2} \quad ...(2)$$

where $\sigma_r^2 = \sigma_{rx}^2 + \sigma_{ry}^2 + \sigma_{rz}^2$ ...(3)

The direction cosines of the shear stress may be determined as follows :

Let a_{sx}, a_{sy}, a_{sz} be the direction cosines of τ_{ns}.

Now $\sigma_{rx} = \sigma_n a_{nx} + \tau_{ns} a_{sx}$

$$a_{sx} = \frac{1}{\tau_{ns}}[\sigma_{rx} - \sigma_n a_{nx}]$$

Similarly

$$a_{sy} = \frac{1}{\tau_{ns}}[\sigma_{ry} - \sigma_n a_{ny}]$$

$$a_{sz} = \frac{1}{\tau_{ns}}[\sigma_{rz} - \sigma_n a_{nz}]$$

1.2.2 Principal Stresses

By considering the forces acting on an elementary tetrahedron, it can be shown that the principal stresses are the roots of the cubic equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

where $I_1 = \sigma_x + \sigma_y + \sigma_z$ is the first invariant of stress.

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$

is the second invariant of stress.

$$I_3 = \sigma_x\sigma_y\sigma_z - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}$$

is the third invariant of stress.

The cubic equation can be solved by a hit and trial procedure or by the **Newton-Raphson method.**

1.2.3 Principal Directions

For the three principal stresses σ_1, σ_2 and σ_3 the principal directions may be determined as follows :

For σ_1 stress,

Let

$$A_1 = \begin{vmatrix} \sigma_y - \sigma_1 & \tau_{yz} \\ \tau_{yz} & \sigma_z - \sigma_1 \end{vmatrix}$$

$$B_1 = -\begin{vmatrix} \tau_{xy} & \tau_{yz} \\ \tau_{xz} & \sigma_z - \sigma_1 \end{vmatrix}$$

$$C_1 = \begin{vmatrix} \tau_{xy} & \sigma_y - \sigma_1 \\ \tau_{xz} & \tau_{yz} \end{vmatrix}$$

Then

$$\sigma_{nx1} = \frac{A_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}}$$

$$\sigma_{ny1} = \frac{B_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}}$$

$$\sigma_{nz1} = \frac{C_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}}$$

Similarly, direction cosines of other stresses can be determined.

An *octahedral plane* is equally inclined to the three co-ordinate axes. The stresses acting on this plane are called the octadedral stresses. This plane has the direction cosines each equal to $\pm\frac{1}{\sqrt{3}}$.

1.3 STRAIN TENSOR : 3D STRAIN

In the case of three-dimensional co-ordinate system, the components of strain are expressed by the strain matrix

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix} \text{ or } \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix}$$

Such that $\gamma_{ij} = 2\varepsilon_{ij}$, $ij = x, y, z$ (where ε_{ij} are the linear strains and γ_{ij} are the shearing strains in terms of the change in the right angle between two orthogonal line elements after deformation).

Proof : Let u, v and w be the displacements in x, y and z direction respectively. For a three dimensional case there are 6 strain components :

$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}$ and γ_{zx}. The three linear strain components are defined by

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \ldots(4)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad \ldots(5)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \quad \ldots(6)$$

Now consider a plane lamina of size dx, dy in the x-y plane (Fig. 1.3). The lines OA and OB, originally orthogonal to each other, are displaced to positions $O'A'$ and $O'B'$ respectively. The shearing strain is equal to the change in the angle at O.

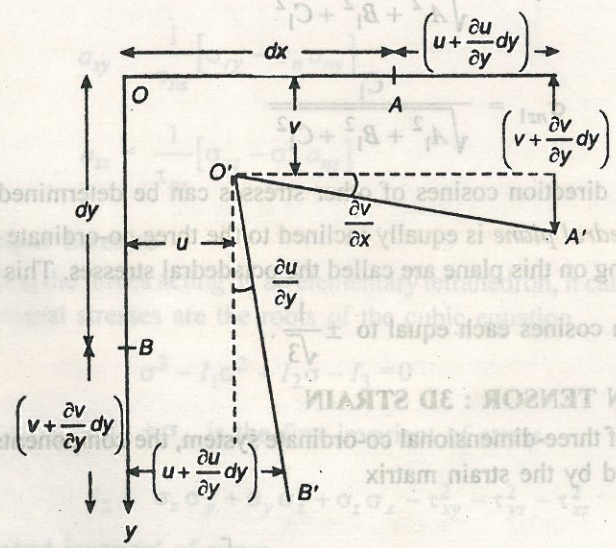


Fig. 1.3

$$\text{Displacement of } A \text{ in } x\text{-direction} = u + \frac{\partial u}{\partial x}.dx$$

$$\text{Displacement of } B \text{ in } y\text{-direction} = v + \frac{\partial v}{\partial y}.dy$$

$$\text{Displacement of } B \text{ in } x\text{-direction} = u + \frac{\partial u}{\partial y}.dy$$

$\therefore$ Total change in the angle at O

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \ldots(7)$$

Similarly

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad \ldots(8)$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \ldots(9)$$

Since linear strain of a diagonal is equal to half the shearing strain, hence

$$\left.\begin{aligned} \varepsilon_{xy} &= \frac{1}{2}\gamma_{xy} \\ \varepsilon_{yz} &= \frac{1}{2}\gamma_{yz} \\ \varepsilon_{zx} &= \frac{1}{2}\gamma_{zx} \end{aligned}\right\} \quad \ldots(10)$$

Hence, the *strain tensor*, consisting of nine strain components can be represented as under :

$$\begin{pmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{pmatrix} \text{ or } \begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{pmatrix}$$

1.3.1 Normal and shearing strains

Suppose on a plane the strain components are ε_{ij}. The direction cosines of the normal to this plane are a_{nx}, a_{ny} and a_{nz}. The normal and shearing strains on this plane can be determined as follows :

$$\varepsilon_{rx} = \varepsilon_x a_{nx} + \frac{\gamma_{xy}}{2}a_{ny} + \frac{\gamma_{xz}}{2}a_{nz}$$

$$\varepsilon_{ry} = \frac{\gamma_{xy}}{2}a_{nx} + \varepsilon_y a_{ny} + \frac{\gamma_{yz}}{2}a_{nz}$$

$$\varepsilon_{rz} = \frac{\gamma_{xy}}{2} a_{nx} + \frac{\gamma_{yz}}{2} a_{ny} + \varepsilon_z a_{nz}$$

Resultant strain

$$\varepsilon_r = \sqrt{\varepsilon_{rx}^2 + \varepsilon_{ry}^2 + \varepsilon_{rz}^2} \quad \text{...(11)}$$

Normal strain

$$\varepsilon_n = \varepsilon_{rx} a_{nx} + \varepsilon_{ry} a_{ny} + \varepsilon_{rz} a_{nz} = \varepsilon_x a_{nx}^2 + \varepsilon_y a_{ny}^2 + \varepsilon_z a_{nz}^2$$

$$+\gamma_{xy} a_{nx} a_{ny} + \gamma_{yz} a_{ny} a_{nz} + \gamma_{xz} a_{nx} a_{nz} \quad \text{...(12)}$$

Shearing strain

$$\gamma_{ns} = 2\sqrt{\varepsilon_r^2 - \varepsilon_n^2} \quad \text{...(13)}$$

The direction cosines of the shearing strain may be determined from

$$\left.\begin{aligned} a_{sx} &= \frac{2}{\gamma_{ns}}\left[\varepsilon_{rx} - \varepsilon_n a_{nx}\right] \\ a_{sy} &= \frac{2}{\gamma_{ns}}\left[\varepsilon_{ry} - \varepsilon_n a_{ny}\right] \\ a_{sz} &= \frac{2}{\gamma_{ns}}\left[\varepsilon_{rz} - \varepsilon_n a_{nz}\right] \end{aligned}\right\} \quad \text{...(14)}$$

1.3.2 Principal Strains

The principal strains are the root of the cubic equation

$$\varepsilon^3 - J_1\varepsilon^2 + J_2\varepsilon - J_3 = 0$$

where $J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$ is the first invariant of strain

$J_2 = \varepsilon_x \varepsilon_y + \varepsilon_y \varepsilon_z + \varepsilon_x \varepsilon_z - \frac{1}{4}\left(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2\right)$ is the second invariant of strain

$J_3 = \varepsilon_x \varepsilon_y \varepsilon_z - \frac{\varepsilon_x \gamma_{yz}^2}{4} - \frac{\varepsilon_y \gamma_{xz}^2}{4} - \frac{\varepsilon_z \gamma_{xy}^2}{4} + \frac{\gamma_{xy}\gamma_{yz}\gamma_{xz}}{4}$ is the third invariant of strain. The roots of equation may be determined by hit and trial method or Newton-Raphson method.

1.3.3 Principal Strain Directions

The principal strain directions may be determined as follows :

For ε_1, Let

$$A_1 = \begin{vmatrix} \varepsilon_y - \varepsilon_1 & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{yz}}{2} & \varepsilon_z - \varepsilon_1 \end{vmatrix}$$

$$B_1 = -\begin{vmatrix} \frac{\gamma_{xy}}{2} & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \varepsilon_z - \varepsilon_1 \end{vmatrix}$$

$$C_1 = \begin{vmatrix} \frac{\gamma_{xy}}{2} & \varepsilon_y - \varepsilon_1 \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} \end{vmatrix}$$

then

$$\left.\begin{aligned} a_{nx1} &= \frac{A_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} \\ a_{ny1} &= \frac{B_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} \\ a_{nz1} &= \frac{C_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} \end{aligned}\right\} \quad \text{...(15)}$$

Similarly, directions of ε_2 and ε_3 may be determined.

1.4 RELATION IN BETWEEN THREE DIMENSIONAL STRESS AND STRAIN (GENERALISED HOOKE'S LAW)

Hooke's Law states that within the elastic limit, stress is proportional to strain. Following are the *generalised Hooke's law equations*. These were given by Navier and Stokes. These equations connect the strain components to stress components :

$$\varepsilon_x = C_{11}\sigma_x + C_{12}\sigma_y + C_{13}\sigma_z + C_{14}\tau_{xy} + C_{15}\tau_{yz} + C_{16}\tau_{zx} \quad \text{...(16a)}$$

$$\varepsilon_y = C_{21}\sigma_x + C_{22}\sigma_y + C_{23}\sigma_z + C_{24}\tau_{xy} + C_{25}\tau_{yz} + C_{26}\tau_{zx} \quad \text{...(16b)}$$

$$\varepsilon_z = C_{31}\sigma_x + C_{32}\sigma_y + C_{33}\sigma_z + C_{34}\tau_{xy} + C_{35}\tau_{yz} + C_{36}\tau_{zx} \quad \text{...(16c)}$$

$$\gamma_{xy} = C_{41}\sigma_x + C_{42}\sigma_y + C_{43}\sigma_z + C_{44}\tau_{xy} + C_{45}\tau_{yz} + C_{46}\tau_{zx} \quad \text{...(16d)}$$

$$\gamma_{yz} = C_{51}\sigma_x + C_{52}\sigma_y + C_{53}\sigma_z + C_{54}\tau_{xy} + C_{55}\tau_{yz} + C_{56}\tau_{zx} \quad \text{...(16e)}$$

$$\gamma_{zx} = C_{61}\sigma_x + C_{62}\sigma_y + C_{63}\sigma_z + C_{64}\tau_{xy} + C_{65}\tau_{yz} + C_{66}\tau_{zx} \quad \text{...(16f)}$$

These equations contain *36 elastic constants*. These elastic constants are independent of the stress components. At a point for an elastic body to be homogeneous, these 36 elastic constants should be same at all points within a region. A point in a material is called *isotropic* if its elastic constants are the same in all directions at the point.

By a series of rotations of axes, it can be shown that the number of independent elastic constants for a homogeneous, isotropic body are only 2. The considerations of homogeneity restrict the total elastic constants within a region to a finite value of 36, further the considerations of isotropy reduce these constants to only 2.

The generalised Hooke's law equations then reduce to

$$\left.\begin{aligned} \varepsilon_x &= C_{11}\sigma_x + C_{12}(\sigma_y + \sigma_z) \\ \varepsilon_y &= C_{11}\sigma_y + C_{12}(\sigma_z + \sigma_x) \\ \varepsilon_z &= C_{11}\sigma_z + C_{12}(\sigma_x + \sigma_y) \\ \gamma_{xy} &= 2(C_{11} - C_{12})\tau_{xy} \\ \gamma_{yz} &= 2(C_{11} - C_{12})\tau_{yz} \\ \text{and} \quad \gamma_{zx} &= 2(C_{11} - C_{12})\tau_{zx} \end{aligned}\right\} \quad ...(17)$$

To calculate the values of C_{11} and C_{12}, let us take

$$\sigma_y = \sigma_z = 0 \quad \text{(uniaxial stress in } x \text{ direction)}$$

$$\Rightarrow \quad C_{11} = \frac{\varepsilon_x}{\sigma_x} = \frac{1}{E}$$

where E is Young's modulus of elasticity.

Also $\quad \varepsilon_y = \varepsilon_z = C_{12}\,\sigma_x = C_{12}\,E\,\varepsilon_x$

$$\therefore \quad C_{12} = \frac{1}{E}\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\mu}{E}$$

where $\mu = \dfrac{1}{m}$ = Poisson's ratio.

Substituting these in Eqn. (17), we get the final equations as follows

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \mu(\sigma_y + \sigma_z)\right] \quad ...(18)$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \mu(\sigma_z + \sigma_x)\right] \quad ...(19)$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu(\sigma_x + \sigma_y)\right] \quad ...(20)$$

$$\gamma_{xy} = \frac{2(1+\mu)}{E}\tau_{xy} \quad ...(21)$$

$$\gamma_{yz} = \frac{2(1+\mu)}{E}\tau_{yz} \quad ...(22)$$

$$\gamma_{zx} = \frac{2(1+\mu)}{E}\tau_{zx} \quad ...(23)$$

1.5 EQUILIBRIUM EQUATIONS

Suppose we have an elemental volume of size dx, dy and dz with the nine stress components acting at the centre of the element (Fig. 1.4). The stress on each face will be equal to the stress at the centre increased or reduced by the distance from the centre to the face times the spatial derivative of the stress.

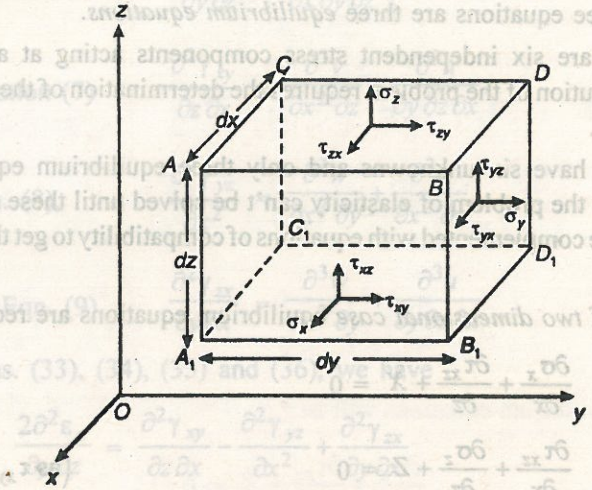


Fig. 1.4

If X, Y and Z denote the components of body forces per unit volume, in the three corresponding directions, then the equation of equilibrium obtained by summing all the forces acting on the element in the x-direction is :

$$\left\{\left(\sigma_x + \frac{\partial\sigma_x}{\partial x}\frac{dx}{2}\right)dy\,dz - \left(\sigma_y - \frac{\partial\sigma_x}{\partial x}\frac{dx}{2}\right)dy\,dz\right\}$$

$$+\left\{\left(\tau_{yx} + \frac{\partial\tau_{yx}}{\partial y}\frac{dy}{2}\right)dx\,dz - \left(\tau_{yx} - \frac{\partial\tau_{yx}}{\partial y}\frac{dy}{2}\right)dx\,dz\right\}$$

$$+\left\{\left(\tau_{zx} + \frac{\partial\tau_{zx}}{\partial z}\frac{dz}{2}\right)dx\,dy - \left(\tau_{zx} - \frac{\partial\tau_{zx}}{\partial z}\frac{dz}{2}\right)dx\,dy\right\}$$

$$+X\,dx\,dy\,dz = 0$$

Dividing all the terms by *dx, dy, dz* and simplifying

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0 \qquad ...(24)$$

In the same way we can get two more equations i.e.

$$\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+\frac{\partial\tau_{xy}}{\partial x}+Y=0 \qquad ...(25)$$

and
$$\frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+Z=0 \qquad ...(26)$$

These three equations are three *equilibrium equations.*

As there are six independent stress components acting at a point, the complete solution of the problem requires the determination of these six stress components.

Since we have six unknowns and only three equilibrium equations are available, so the problem of elasticity can't be solved until these equilibrium equations are complemented with equations of compatibility to get the complete solution.

In case of *two dimensional case* equilibrium equations are reduced to

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}+X=0$$

and
$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}+Z=0 \qquad \left(\text{as } \tau_{xy}=\tau_{yz}=0\right)$$

1.6 COMPATIBILITY EQUATIONS

The equations resulting from the application of strain equations are known as the *compatibility equations*. These are known as *Saint Venant's equations* as well.

Differentiating Eqn. (4) twice with respect to *y*, Eqn. (5) twice with respect to *x* and Eqn. (6) once with respect to *x* and then with respect to *y*, we get

$$\frac{\partial^2\varepsilon_x}{\partial y^2}=\frac{\partial^3 u}{\partial x\,\partial y^2} \qquad ...(27)$$

$$\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^3 v}{\partial y\,\partial x^2} \qquad ...(28)$$

$$\frac{\partial^2\gamma_{xy}}{\partial x\,\partial y}=\frac{\partial^3 u}{\partial x\,\partial y^2}+\frac{\partial^3 v}{\partial x^2\,\partial y} \qquad ...(29)$$

By these three equations we get

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_z}{\partial x^2}=\frac{\partial^2\gamma_{yz}}{\partial x\,\partial y} \qquad ...(30)$$

Similarly
$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\,\partial z} \qquad ...(31)$$

and
$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\,\partial x} \qquad ...(32)$$

From equation (4)
$$\frac{\partial^2\varepsilon_x}{\partial y\,\partial z}=\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} \qquad ...(33)$$

From equation (7)
$$\frac{\partial^2\gamma_{xy}}{\partial z\,\partial x}=\frac{\partial^3 v}{\partial x^2\,\partial z}+\frac{\partial^3 u}{\partial y\,\partial z\,\partial x} \qquad ...(34)$$

From Eqn. (8),
$$\frac{\partial^2\gamma_{yz}}{\partial x^2}=\frac{\partial^3 w}{\partial x^2\,\partial y}+\frac{\partial^3 v}{\partial x^2\,\partial z} \qquad ...(35)$$

and from Eqn. (9)
$$\frac{\partial^2\gamma_{zx}}{\partial y\,\partial x}=\frac{\partial^3 w}{\partial x^2\,\partial y}+\frac{\partial^3 u}{\partial y\,\partial z\,\partial x} \qquad ...(36)$$

From Eqns. (33), (34), (35) and (36), we have

$$\frac{2\partial^2\varepsilon_x}{\partial y\,\partial z}=\frac{\partial^2\gamma_{xy}}{\partial z\,\partial x}-\frac{\partial^2\gamma_{yz}}{\partial x^2}+\frac{\partial^2\gamma_{zx}}{\partial y\,\partial x}$$

or
$$\frac{2\partial^2\varepsilon_x}{\partial y\,\partial z}=\frac{\partial}{dx}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right) \qquad ...(37)$$

Similarly,
$$\frac{2\partial^2\varepsilon_y}{\partial z\,\partial x}=\frac{\partial}{dy}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right) \qquad ...(38)$$

and
$$\frac{2\partial^2\varepsilon_z}{\partial x\,\partial y}=\frac{\partial}{dz}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right) \qquad ...(39)$$

Equations (30), (31), (32), (37), (38) and (39) are the *six compatibility equations.*

1.6.1 Compatibility Equation in Two Dimensional Case

Suppose in two dimensional case the strain components ε_y, γ_{xy} and γ_{yz} are zero. The other strain components $\varepsilon_x, \varepsilon_z$ and γ_{xz} may be given by the following Hooke's law equations :

$$\varepsilon_y = 0 = \frac{1}{E}[\sigma_y - \mu(\sigma_z + \sigma_x)]$$

$$\therefore \quad \sigma_y = \mu(\sigma_z + \sigma_x) \qquad ...(40)$$

$$\therefore \quad \varepsilon_x = \frac{1}{E}[\sigma_x - \mu\sigma_y - \mu\sigma_z]$$

$$= \frac{1}{E}[\sigma_x - \mu\sigma_z - \mu^2(\sigma_x + \sigma_z)]$$

$$\Rightarrow \quad \varepsilon_x = \frac{1-\mu^2}{E}\left[\sigma_x - \frac{\mu}{1-\mu}\sigma_z\right] \qquad ...(41)$$

Similarly,

$$\varepsilon_z = \frac{1-\mu^2}{E}\left[\sigma_z - \frac{\mu}{1-\mu}\sigma_x\right] \qquad ...(42)$$

and

$$\gamma_{xz} = \frac{2(1+\mu)}{E}\tau_{xz} \qquad ...(43)$$

$$\gamma_{xy} = \gamma_{yz} = 0$$

$$\tau_{xy} = \tau_{yz} = 0$$

The equilibrium equations will be

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \qquad ...(44)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \qquad ...(45)$$

For two dimensional case, the six compatibility equations (30, 31, 32, 37, 38 and 39) reduce to one equation

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \qquad ...(46)$$

(*i*) **Plane stress case.** Substituting the values of $\varepsilon_x, \varepsilon_z$ and γ_{xz} from the Hooke's law equations [(40) to (43)]

$$\frac{\partial^2}{\partial z^2}(\sigma_x - \mu\sigma_z) + \frac{\partial^2}{\partial x^2}(\sigma_z - \mu\sigma_z) = 2(1+\mu)\frac{\partial^2 \tau_{xz}}{\partial x \partial z} \qquad ...(47)$$

Differentiating Eqn. (44) with respect to x and the Eqn. (45) with respect to z and adding them together, we get

$$\frac{2\partial^2 \tau_{xz}}{\partial x \partial z} = -\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_z}{\partial z^2}\right) - \left(\frac{\partial X}{\partial x} + \frac{\partial Z}{\partial z}\right) \qquad ...(48)$$

On simplification,

putting the value of $\frac{2\partial^2 \tau_{xz}}{\partial x \partial z}$ in Eqn. (47), we get

$$\frac{\partial^2}{\partial z^2}(\sigma_x - \mu\sigma_z) + \frac{\partial^2}{\partial x^2}(\sigma_z - \mu\sigma_x) + (1+\mu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_z}{\partial z^2}\right)$$

$$+ (1+\mu)\left(\frac{\partial X}{\partial x} + \frac{\partial Z}{\partial z}\right) = 0$$

Simplifying

$$\frac{\partial^2 \sigma_x}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_z}{\partial z^2} = -(1+\mu)\left(\frac{\partial X}{\partial x} + \frac{\partial Z}{\partial z}\right)$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)(\sigma_x + \sigma_z) = -(1+\mu)\left(\frac{\partial X}{\partial x} + \frac{\partial Z}{\partial z}\right) \qquad ...(49)$$

If body forces are absent or constant then we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)(\sigma_x + \sigma_z) = 0 \qquad ...(50)$$

which is the required compatibility equation in terms\of stress for the plane stress case.

(*ii*) **Plane strain case.** Putting the value of $\varepsilon_x, \varepsilon_z$ and γ_{xz} from Eqn. (41), (42) and (43), into equation (46), we get

$$(1-\mu^2)\frac{\partial^2 \sigma_x}{\partial z^2} - \mu(1+\mu)\frac{\partial^2 \sigma_z}{\partial z^2} + (1-\mu^2)\frac{\partial^2 \sigma_y}{\partial x^2} - \mu(1+\mu)\frac{\partial^2 \sigma_x}{\partial x^2}$$

$$= 2(1+\mu)\frac{\partial^2 \tau_{xz}}{\partial x \partial z}$$

Substituting the value of $\frac{2\partial^2 \tau_{xz}}{\partial x \partial z}$ from Eqn. (48)

$$\left(1-\mu^2\right)\frac{\partial^2\sigma_x}{\partial z^2}-\mu(1+\mu)\frac{\partial^2\sigma_z}{\partial z^2}+\left(1-\mu^2\right)\frac{\partial^2\sigma_z}{\partial x^2}-\mu(1+\mu)\frac{\partial^2\sigma_x}{\partial x^2}$$

$$+(1+\mu)\left[\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_z}{\partial z^2}+\frac{\partial X}{\partial x}+\frac{\partial Z}{\partial z}\right]=0,$$

On simplification,

$$\frac{\partial^2\sigma_x}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial x^2}=-\frac{1}{1-\mu}\left(\frac{\partial X}{\partial x}+\frac{\partial Z}{\partial z}\right)$$

or $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma_x+\sigma_z)=-\frac{1}{1-\mu}\left(\frac{\partial X}{\partial x}+\frac{\partial Z}{\partial z}\right) \quad ...(51)$$

If the body forces are absent or constant, we have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma_x+\sigma_z)=0 \quad ...(52)$$

This equation is the same as equation (50) found for the plane stress case. Thus in case of constant body forces (or no body forces) same compatibility equation holds both for the cases of plane stress and for case of plane strain.

1.7 AIRY'S STRESS FUNCTION

The solution of a two dimensional problem of elasticity is possible by using the differential equations of equilibrium together with the compatibility equations. These equations are

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}=0 \quad ...(53)$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}+Z=0 \quad ...(54)$$

and $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma_x+\sigma_z)=0 \quad ...(55)$$

(where Z is the body force per unit volume in the Z-direction. It is assumed that the weight of the body is directed towards Z-axis, so that X is zero)

Above three equations, when reduced into one single equation, a stress function ϕ is introduced.

$$\sigma_x=\frac{\partial^2\phi}{\partial z^2} \quad ...(56)$$

$$\sigma_z=\frac{\partial^2\phi}{\partial x^2} \quad ...(57)$$

and $$\tau_{xz}=-\frac{\partial^2\phi}{\partial x\,\partial z}-Z.x \quad ...(58)$$

This function ϕ was introduced by G.B. Airy, hence it is called as *Airy's Stress Function.*

Differentiating equations (56), (57) and (58)

$$\frac{\partial\sigma_x}{\partial x}=\frac{\partial^3\phi}{\partial x\,\partial z^2},\quad \frac{\partial\tau_{xz}}{\partial z}=-\frac{\partial^3\phi}{\partial x\,\partial z^2}$$

$$\frac{\partial\sigma_z}{\partial z}=\frac{\partial^3\phi}{\partial z\,\partial x^2},\quad \frac{\partial\tau_{xz}}{\partial x}=-\frac{\partial^3\phi}{\partial x^2\,\partial z}-Z$$

$$\therefore\quad \frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}=\frac{\partial^3\phi}{\partial x\,\partial z^2}-\frac{\partial^3\phi}{\partial x\,\partial z^2}=0$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}+Z=-\frac{\partial^3\phi}{\partial x^2\,\partial z}-Z+\frac{\partial^3\phi}{\partial z\,\partial x^2}+Z=0$$

Substituting the proper derivatives of stress components as function of Airys Stress Function in the compatibility equation (55), we get

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\,\partial z^2}+\frac{\partial\phi^4}{\partial z^4}=0$$

or $$\nabla^2\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}\right]=0$$

or $$\boxed{\nabla^4\phi=0} \quad ...(60)$$

This is a biharmonic differential equation of fourth order and is known as the compatibility equation in terms of Airy's Stress Function.

Example 1.1 The state of stress at a point is given by $\sigma_x=-80$, $\sigma_y=25$, $\sigma_z=-35$, $\tau_{xy}=-16$, $\tau_{yz}=-30$ and $\tau_{xz}=25$ MN/m². Determine the normal and shearing stresses on a plane perpendicular to the x-axis whose direction cosines are $a_{nx}=1/4$, $a_{ny}=1/2$.

Solution. $a_{nx}^2+a_{ny}^2+a_{nz}^2=1$

$$\therefore\quad a_{nz}^2=1-\frac{1}{16}-\frac{1}{4}=\frac{11}{16}\quad\Rightarrow\quad a_{nz}=0.829$$

$$\sigma_{rx} = \sigma_x a_{nx} + \tau_{xy} + \tau_{xz} a_{nz}$$

$$= -80 \times \frac{1}{4} + 16 \times \frac{1}{2} + 25 \times 0.829 = 8.729 \text{ MPa}$$

$$\sigma_{ry} = \tau_{xy} a_{nx} + \sigma_y a_{ny} + \tau_{yz} a_{nz}$$

$$= 16 \times \frac{1}{4} + 25 \times \frac{1}{2} - 30 \times 0.829 = -8.375 \text{ MPa}$$

$$\sigma_{rz} = \tau_{xz} a_{nx} + \tau_{yz} a_{ny} + \sigma_z a_{nz}$$

$$= 25 \times \frac{1}{4} + 30 \times \frac{1}{2} - 35 \times 0.829 = -37.770 \text{ MPa}$$

Resultant stress $\sigma_r = \sqrt{\sigma_{rx}^2 + \sigma_{ry}^2 + \sigma_{rz}^2}$

$$= \left\{(8.728)^2 + (-8.375)^2 + (-37.770)^2\right\}^{1/2}$$

$$= 39.66 \text{ MN/m}^2$$

Normal stress $\sigma_n = \sigma_{rx} a_{nx} + \sigma_{ry} a_{ny} + \sigma_{rz} a_{nz}$

$$= 8.729 \times \frac{1}{4} - 8.375 \times \frac{1}{2} - 37.77 \times 0.829$$

$$= -33.317 \text{ MN/m}^2$$

Shearing stress $\tau_{ns} = \sqrt{\sigma_r^2 - \sigma_n^2} = \sqrt{1572.909 - 1110.02}$

$$= 21.515 \text{ MN/m}^2$$

Example 1.2 The state of stress at a point is specified by the following stress components, $\sigma_x = 70$, $\sigma_y = 10$, $\sigma_z = -20$, $\tau_{xy} = -40$, $\tau_{yz} = \tau_{xz} = 0$ MN/m^2. Determine the principal stresses and the direction cosines of the maximum principal stress.

Solution. $I_1 = \sigma_x + \sigma_y + \sigma_z = 70 + 10 - 20 = 60$

$$I_2 = \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_x \sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$

$$= 70 \times 10 - 10 \times 20 - 70 \times 20 - (-40)^2 - 0 - 0$$

$$= -2500$$

$$I_3 = \sigma_x \sigma_y \sigma_z - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2 + 2\tau_{xy} \tau_{yz} \tau_{xz}$$

$$= -70 \times 10 \times 20 - 70 \times 0 - 10 \times 0 + 20 \times 1600 + 0$$

$$= 18000$$

$$\therefore \quad \sigma^3 - 60\sigma^2 - 2500\sigma - 18000 = 0$$

By hit and trial we find that $\sigma_1 = 90$ MN/m^2 is one of the roots. The other two roots are -10 and -20 MN/m^2.

hence $\sigma_1 = 90$ MN/m^2, $\sigma_2 = -10$ MN/m^2

$$\sigma_3 = -20 \text{ MN/m}^2$$

The direction cosines of σ_1 may be determined as follows

$$A_1 = \begin{vmatrix} -80 & 0 \\ 0 & -110 \end{vmatrix} = 8800$$

$$B_1 = -\begin{vmatrix} -40 & 0 \\ 0 & -110 \end{vmatrix} = -4400$$

$$C_1 = \begin{vmatrix} -40 & -80 \\ 0 & 0 \end{vmatrix} = 0$$

$$\sqrt{A_1^2 + B_1^2 + C_1^2} = 9838.7$$

$$\therefore \quad a_{nx1} = \frac{8800}{9838.7} = 0.8944$$

$$a_{ny1} = \frac{-4400}{9838.7} = -0.4472$$

$$a_{nz1} = 0$$

Example 1.3 The principal stresses at a point are 330 MN/m^2, 50 MN/m^2 and -120 MN/m^2. Determine the octahedral normal and shearing stresses.

Solution. The direction cosines of the octahedral plane are $a_{nx} = a_{ny} = a_{nz} = 1/\sqrt{3}$

Normal stress $\sigma_n = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$

$$= \frac{1}{3}(330 + 50 - 120) = 86.67 \text{ MN/m}^2$$

Resultant stress $\sigma_r = \sqrt{\frac{1}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)}$

$= \sqrt{\frac{1}{3}\left(330^2 + 50^2 + 120^2\right)} = 204.78 \text{ MN/m}^2$

Shear stress $\tau_{ns} = \sqrt{\sigma_r^2 - \sigma_n^2} = \sqrt{204.78^2 - 86.67^2}$

$= 185.52 \text{ MN/m}^2$

Example 1.4 At a point in a stressed material the stress components are

$\sigma_x = -40, \ \sigma_y = 80, \ \sigma_z = 120,$

$\tau_{xy} = 72, \ \tau_{yz} = 46, \ \tau_{xz} = 32 \text{ MPa}$

Calculate the normal, shear and resultant stresses on a plane whose normal makes an angle of 48° with the x-axis and 61° with the y-axis.

Solution. $a_{nx} = \cos 48° = 0.66913$

$a_{ny} = \cos 61° = 0.48481$

$a_{nz} = \sqrt{1 - a_{nx}^2 - a_{ny}^2} = \sqrt{1 - 0.44773 - 0.23504} = 0.56323$

$\sigma_{rx} = -40 \times 0.66913 + 72 \times 0.48481 + 32 \times 0.65323$

$= 26.17 \text{ MPa}$

$\sigma_{ry} = 72 \times 0.66913 + 80 \times 0.48481 + 46 \times 0.56323$

$= 112.87 \text{ MPa}$

$\sigma_{rz} = 32 \times 0.66913 + 46 \times 0.48481 + 120 \times 0.56323$

$= 111.29 \text{ MPa}$

Resultant stress

$\sigma_n = \sqrt{\sigma_{rx}^2 + \sigma_{ry}^2 + \sigma_{rz}^2}$

$= \sqrt{(26.17)^2 + (112.87)^2 + (111.29)^2}$

$= 160.65 \text{ MPa}$

Normal stress $\sigma_n = 26.17 \times 0.66913 + 112.87 \times 0.48481$

$+ 111.29 \times 0.56323$

$= 134.91 \text{ MPa}$

Shear stress $\tau_{ns} = \sqrt{\sigma_r^2 - \sigma_n^2} = \sqrt{25809 - 18200}$

$= 87.23 \text{ MPa}$

Example 1.5 In a triaxial stress system, the six components of the stress at a point are given below :

$\sigma_x = 6 \text{ MN/m}^2, \quad \tau_{xy} = \tau_{yx} = 1 \text{ MN/m}^2$

$\sigma_y = 5 \text{ MN/m}^2 \quad \tau_{yz} = \tau_{zy} = 3 \text{ MN/m}^2$

$\sigma_z = 4 \text{ MN/m}^2 \quad \tau_{zx} = \tau_{xz} = 2 \text{ MN/m}^2$

Find the magnitude of three principal stresses. 6 Marks (UPTU 2002-03)

Solution. $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

where $I = \sigma_x + \sigma_y + \sigma_z$

$= 6 + 5 + 4 = 15 \text{ MN/m}^2$

$I_2 = \sigma_x \sigma_y + \sigma_{yz} + \sigma_y \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$

$= 6 \times 5 + 5 \times 4 + 4 \times 6 - 1^2 - 3^2 - 2^2 = 60$

$I_3 = \sigma_x \sigma_y \sigma_z - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}$

$= 6 \times 5 \times 4 - 6 \times 3^2 - 5 \times 2^2 - 4 \times 1^2 + 2 \times 1 \times 3 \times 2$

$= 54$

$\therefore \quad \sigma^3 - 15\sigma^2 + 60\sigma - 54 = 0$

Solving, $\sigma_1 = 9 \text{ MN/m}^2$

$\sigma_2 = 4.732 \text{ MN/m}^2$

$\sigma_3 = 1.248 \text{ MN/m}^2$

Example 1.6 At a point P in a body, $\sigma_x = 30 \text{ kN/cm}^2$, $\sigma_y = -10 \text{ kN/cm}^2$ and $\sigma_z = +10 \text{ kN/cm}^2$ and $\tau_{xy} = \tau_{yz} = \tau_{zx} = 10 \text{ kN/cm}^2$. Determine the normal and shearing stresses on a plane that is equally inclined to all the three axes.

6 Marks (UPTU 2001-02)

Solution. A plane that is equally inclined to all the three axes will have

$a_{nx} = a_{ny} = a_{nz} = \frac{1}{\sqrt{3}} \quad$ since $n_x^2 + n_y^2 + n_z^2 = 1$

Normal stress $\sigma_n = \sigma_x a_{nx}^2 + \sigma_y a_{ny}^2 + \sigma_z a_{nz}^2 + 2\left(\tau_{xy} a_{nx} a_{ny} \right.$

$\left. \tau_{yz} a_{ny} a_{nz} + \tau_{xz} a_{nx} a_{nz}\right)$

$= \frac{1}{3}(30 - 10 + 10 + 20 + 20 + 20) = 30 \text{ kN/cm}^2$

$$\sigma_{rx} = \sigma_x a_{nx} + \tau_{xy} a_{ny} + \tau_{xz} a_{nz}$$

$$= \frac{1}{\sqrt{3}}(30+10+10) = \frac{50}{\sqrt{3}}$$

$$\sigma_{ry} = \tau_{xy} a_{nx} + \sigma_y a_{ny} + \tau_{yz} a_{nz}$$

$$= \frac{1}{\sqrt{3}}(10-10+10) = \frac{30}{\sqrt{3}}$$

$$\sigma_{rz} = \tau_{xz} a_{nx} + \tau_{yz} a_{ny} + \sigma_z a_{nz}$$

$$= \frac{1}{\sqrt{3}}(10+10+10) = \frac{30}{\sqrt{3}}$$

$$\sigma_r = \sqrt{\sigma_{rx}^2 + \sigma_{ry}^2 + \sigma_{rz}^2} = \sqrt{\frac{2500}{3} + \frac{100}{3} + \frac{900}{3}}$$

$$= \sqrt{\frac{3500}{3}}$$

$$\tau_{ns} = \sqrt{\sigma_r^2 - \sigma_n^2} = \sqrt{\frac{3500}{3} - 900} = 20\sqrt{\frac{2}{3}}$$

Example 1.7 The state of stress at a point is characterised by the components

$$\sigma_x = 100 \text{ MPa}, \quad \sigma_y = -40 \text{ MPa}, \quad \sigma_z = 80 \text{ MPa}$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

Determine the maximum values of the shear stresses, their associated normal stresses, the octahedral shear stress and its associated normal stress.

Solution. The given stress components are the principal stresses, since the shear stresses are zero.

Arranging the terms such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, $\sigma_1 = 100$ MPa, $\sigma_2 = 80$ MPa, $\sigma_3 = -40$ MPa

Hence
$$\tau_1 = \frac{\sigma_2 - \sigma_3}{2} = \frac{80+40}{2} = 60 \text{ MPa}$$

$$\tau_2 = \frac{\sigma_3 - \sigma_1}{2} = \frac{-40-100}{2} = -70 \text{ MPa}$$

$$\tau_3 = \frac{\sigma_1 - \sigma_2}{2} = \frac{100-80}{2} = 10 \text{ MPa}$$

The associated normal stresses are

$$\sigma_1^* = \frac{\sigma_2 + \sigma_3}{2} = \frac{80-40}{2} = 20 \text{ MPa}$$

$$\sigma_2^* = \frac{\sigma_3 + \sigma_1}{2} = \frac{-40+100}{2} = 30 \text{ MPa}$$

$$\sigma_3^* = \frac{\sigma_1 + \sigma_2}{2} = \frac{100+80}{2} = 90 \text{ MPa}$$

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

$$= 61.8 \text{ MPa}$$

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{140}{3} = 46.7 \text{ MPa}$$

Example 1.8 The strain components at a point are given by : $\varepsilon_x = 100$, $\varepsilon_y = 50$, $\varepsilon_z = 40$ μ-strains and $\gamma_{xy} = 20$, $\gamma_{yz} = 10$, $\gamma_{xz} = 15$ μ-radians. Calculate the normal and shearing strains on a plane whose normal has the direction cosines $1/\sqrt{3}$, $\sqrt{2/3}$, 0.

Solution.
$$\varepsilon_{rx} = \varepsilon_x a_{nx} + \frac{1}{2}\gamma_{xy} a_{ny} + \frac{1}{2}\gamma_{xz} a_{nz}$$

$$= 100 \times \frac{1}{\sqrt{3}} + \frac{1}{2} \times 20 \times \sqrt{\frac{2}{3}} + 0$$

$$= 65.9 \ \mu\text{-strains}$$

$$\varepsilon_{ry} = \frac{1}{2}\gamma_{xy} a_{nx} + \varepsilon_y a_{ny} + \frac{1}{2}\gamma_{yz} a_{nz}$$

$$= \frac{1}{2} \times 20 \times \frac{1}{\sqrt{3}} + 50 \times \sqrt{\frac{2}{3}} + 0$$

$$= 46.598 \ \mu\text{-strains}$$

$$\varepsilon_{rz} = \frac{1}{2}\gamma_{xz} a_{nx} + \frac{1}{2}\gamma_{yz} a_{ny} + \varepsilon_z a_{nz}$$

$$= \frac{1}{2} \times 15 \times \frac{1}{\sqrt{3}} + \frac{1}{2} \times 10 \times \sqrt{\frac{2}{3}} + 0$$

$$= 8.412 \ \mu\text{-strains}$$

Resultant strain

$$\varepsilon_r = \sqrt{\varepsilon_{rx}^2 + \varepsilon_{ry}^2 + \varepsilon_{rz}^2}$$

$$= 81.147 \ \mu\text{-strains}$$

Normal strain $\varepsilon_n = \varepsilon_{rx} a_{nx} + \varepsilon_{ry} a_{ny} + \varepsilon_{rz} a_{nz}$

$$= 65.9 \times \frac{1}{\sqrt{3}} + 46.598 \times \sqrt{\frac{2}{3}} + 0 = 76.10 \ \mu\text{-strains}$$

Shearing strain $\gamma_{ns} = 2\sqrt{\varepsilon_r^2 - \varepsilon_n^2}$

$$= 2\sqrt{6584.24 - 5791.21} = 56.346 \ \mu\text{-rad}$$

Example 1.9 The stress components at a point are given by $\sigma_x = 20$, $\sigma_y = 10$, $\sigma_z = 15$, $\tau_{xy} = 5$, $\tau_{yz} = 10$, $\tau_{xz} = 20$ MPa. Calculate the strain components, taking $E = 200$ GPa and $\nu = 0.25$.

Solution. $G = \dfrac{E}{2(1+\nu)} = \dfrac{200}{2\,(1+0.25)} = 80$ GPa

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

$$= \frac{10^6}{200 \times 10^9}\left[20 - 0.25(10+15)\right] = 68.75 \times 10^{-6}$$

Similarly $\varepsilon_y = -6.25 \times 10^{-6}$

and $\varepsilon_z = 37.5 \times 10^{-6}$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{5 \times 10^6}{80 \times 10^9} = 62.5 \times 10^{-6}$$

$$\gamma_{yz} = \frac{10 \times 10^6}{80 \times 10^9} = 125 \times 10^{-6}$$

$$\gamma_{xz} = \frac{20 \times 10^6}{80 \times 10^9} = 250 \times 10^{-6}$$

Example 1.10 Given the following Airys stress function $\phi = \dfrac{H}{\pi} z \tan^{-1}\dfrac{x}{z}$. Determine the stress components σ_x, σ_z and τ_{xz}.

Solution. By successive differentiation of the stress function, we get

$$\frac{\partial\phi}{\partial z} = \frac{H}{\pi}\left[-\frac{xz}{x^2+z^2} + \tan^{-1}\frac{x}{z}\right] \quad ...(i)$$

$$\frac{\partial^2\phi}{\partial z^2} = \frac{H}{\pi}\frac{1}{x^2+z^2}\left[2xz^2 - xz^2 - x^3 - xz^2 - x^3\right]$$

$$= -\frac{2H}{\pi}\frac{x^3}{\left(x^2+z^2\right)^2} \quad ...(ii)$$

$$\frac{\partial^3\phi}{\partial z^3} = \frac{H}{\pi}\frac{8x^3 z}{\left(x^2+z^2\right)^3} \quad ...(iii)$$

$$\frac{\partial^4\phi}{\partial z^4} = \frac{H}{\pi}\frac{8x^5 - 40x^3z^2}{\left(x^2+z^2\right)^4} \quad ...(iv)$$

$$\frac{\partial^3\phi}{\partial z^2\,\partial x} = -\frac{2H}{\pi}\frac{3x^2z^2 - x^4}{\left(x^2+z^2\right)^3} \quad ...(v)$$

$$\frac{\partial^4\phi}{\partial z^2\,\partial x^2} = \frac{H}{\pi}\frac{64x^3z^2 - 24xz^4 - 8x^5}{\left(x^2+z^2\right)^4} \quad ...(vi)$$

Similarly

$$\frac{\partial\phi}{\partial x} = \frac{H}{\pi}\frac{z^2}{x^2+z^2} \quad ...(vii)$$

$$\frac{\partial\phi}{\partial x^2} = -\frac{2H}{\pi}\frac{xz^2}{\left(x^2+z^2\right)^2} \quad ...(viii)$$

$$\frac{\partial^2\phi}{\partial x^3} = \frac{2H}{\pi}z^2\frac{3x^2 - z^2}{\left(x^2+z^2\right)^3} \quad ...(ix)$$

$$\frac{\partial^4\phi}{\partial x^4} = \frac{H}{\pi}\frac{24xz^4 - 24x^3z^2}{\left(x^2+z^2\right)^4} \quad ...(x)$$

Now

$$\sigma_x = \frac{\partial^2\phi}{\partial z^2} = -\frac{2H}{\pi}\frac{x^3}{\left(x^2+z^2\right)} \quad \textbf{Ans.}$$

$$\sigma_z = \frac{\partial^2\phi}{\partial x^2} = -\frac{2H}{\pi}\frac{xz^2}{\left(x^2+z^2\right)^2} \quad \textbf{Ans.}$$

$$\tau_{xz} = \frac{\partial^2\phi}{\partial x\,\partial z} = -\frac{2H}{\pi}\frac{x^2 z}{\left(x^2+z^2\right)^2} \quad \textbf{Ans.}$$

USEFUL RESULTS

1. $a_{sx} = \dfrac{1}{\tau_{ns}}[\sigma_{rx} - \sigma_n a_{nx}]$

2. $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$, where

$I_1 = \sigma_x + \sigma_y + \sigma_z$

$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$

$I_3 = \sigma_x\sigma_y\sigma_z - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}$

3. $\sigma_{nx1} = \dfrac{A_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}}$ etc. where $A_1 = \begin{vmatrix} \sigma_y - \sigma_1 & \tau_{yz} \\ \tau_{yz} & \sigma_z - \sigma_1 \end{vmatrix}$ etc.

4. $\gamma_{ij} = 2\varepsilon_{ij}$

5. $\varepsilon_n = \varepsilon_{rx} a_{nx} + \varepsilon_{ry} a_{ny} + \varepsilon_{rz} a_{nz}$

where $\varepsilon_{rx} = \varepsilon_x a_{nx} + \dfrac{\gamma_{xy}}{2} a_{ny} + \dfrac{\gamma_{xz}}{2} a_{nz}$ etc.

6. $\varepsilon_r = \sqrt{\varepsilon_{rx}^2 + \varepsilon_{ry}^2 + \varepsilon_{rz}^2}$ 7. $\gamma_{ns} = 2\sqrt{\varepsilon_r^2 - \varepsilon_n^2}$

8. $a_{sx} = \dfrac{2}{\gamma_{ns}}[\varepsilon_{rx} - \varepsilon_n a_{nx}]$ (similarly for a_{sy}, a_{sz})

9. $\varepsilon^3 - J_1\varepsilon^2 + J_2\varepsilon - J_3 = 0$

where $J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_z - \dfrac{1}{4}\left(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2\right)$

$J_3 = \varepsilon_x\varepsilon_y\varepsilon_z - \dfrac{\varepsilon_x\gamma_{yz}^2}{4} - \dfrac{\varepsilon_y\gamma_{xz}^2}{4} - \dfrac{\varepsilon_z\gamma_{xy}^2}{4} + \dfrac{\gamma_{xy}\gamma_{yz}\gamma_{xz}}{4}$

10. $a_{nx1} = \dfrac{A_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}}$ where $A_1 = \begin{vmatrix} \varepsilon_y - \varepsilon_1 & \dfrac{\gamma_{yz}}{2} \\ \dfrac{\gamma_{yz}}{2} & \varepsilon_z - \varepsilon_1 \end{vmatrix}$ etc.

11. $\varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right]$ (similarly for ε_y and ε_z)

$\gamma_{xy} = \dfrac{2(1+\mu)}{E}\tau_{xy}$ similarly for γ_{yz} and γ_{zx})

12. $\dfrac{\partial^2\varepsilon_x}{\partial z^2} + \dfrac{\partial^2\varepsilon_z}{\partial x^2} = \dfrac{\partial^2\gamma_{xz}}{\partial x\,\partial z}$

REVIEW QUESTIONS

Write short notes on the following :

(*i*) Three dimensional stress

(*ii*) Three dimensional strain

(*iii*) Relationship in three dimensional stresses and strains

(*iv*) Compatibility equations

(*v*) Airy's Function

(*vi*) Generalised Hooke's Law

NUMERICAL PROBLEMS

1. The Cartesian components of stress at a point are given as below

$\sigma_x = 15, \quad \sigma_y = \sigma_z = 8$

$\tau_{xy} = 6 \quad \tau_{yz} = 4 \quad \tau_{xz} = 4$ MPa

Determine the normal and shear stresses on a plane whose direction cosines are $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$. **[Ans.** 19.57 MPa, 3.83 MPa]

2. The Cartesian components of stresses at a point are given as below :

$\sigma_x = 7,\ \sigma_y = 6,\ \sigma_z = 5,\ \tau_{xy} = 2,\ \tau_{yz} = -2,\ \tau_{xz} = 0$ MPa

Determine the values of principal stresses.

[Ans. 9 MPa, 6 MPa, 3 MPa]

3. The state of stress at a point for a given reference xyz is given by the following array of terms :

$$\begin{bmatrix} 15 & 8 & -6 \\ 8 & -12 & 5 \\ -6 & 5 & 8 \end{bmatrix} \text{MPa}$$

Determine the principal stresses.

[Ans. 19.19 MPa, 10.27 MPa, – 10.02 MPa]

4. The principal stresses at a point on a plane are

$\sigma_1 = 50,\ \sigma_2 = 40$ and $\sigma_3 = -20$ MPa.

Determine the normal and shear stresses on this plane if its direction cosines are $\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}$. **[Ans.** 2.14 MPa, 29.8 MPa)

2

Castigliano's Theorem

2.1 INTRODUCTION

Castigliano's theorem is extremely useful in determining the displacements of structures as well as in the solutions of many statically indeterminate structures. Several examples will illustrate this subsequently. Castigliano's theorem gives the displacement of points in the directions of the external forces where they are acting.

It is necessary to know that in developing the theorem of Castigliano, it has been assumed that the elastic body satisfies Hooke's law.

2.2 STATEMENT OF CASTIGLIANO'S THEOREM

This theorem states that *the rate of change of the strain energy with respect to statically independent force gives the component deflection of this force in the direction of the force.*

In more simple words it can be said that if a body is subjected to a number of loads the partial derivative of the total strain energy with respect to any load gives the deflection in the direction of that load.

2.3 CASTIGLIANO'S THEOREM

Proof. Consider a beam with forces P_1, P_2, P_3 etc. acting at points 1, 2, 3 etc. (Fig. 2.1). If x_1, x_2, x_3 etc. are the deflections in the direction of the forces, then the total strain energy of the system is equal to the work done.

$$U = \frac{1}{2}P_1x_1 + \frac{1}{2}P_2x_2 + \frac{1}{2}P_3x_3 + \ldots \qquad \ldots(2.1)$$

If one of the loads, P_1, is now increased by an amount δP_1 the changes in deflections will be δx_1, δx_2, δx_3 etc. as shown in Fig. 2.1.

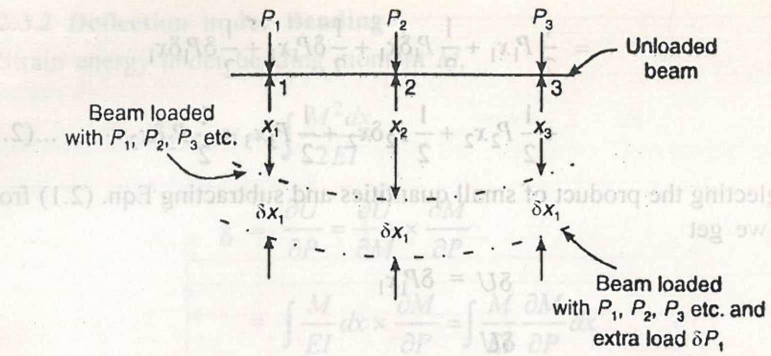


Fig. 2.1

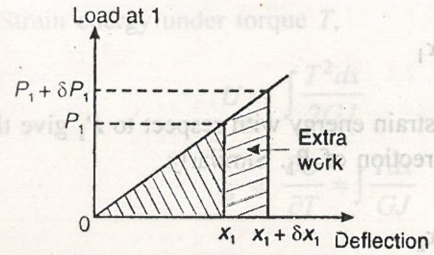


Fig. 2.2

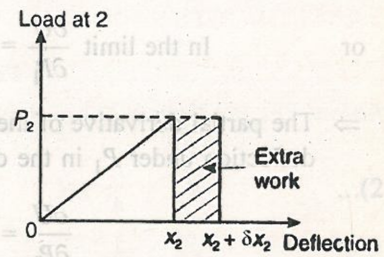


Fig. 2.3

Extra work done at 1, as shown in Fig. 2.2 is

$$= \left(P_1 + \frac{1}{2}\delta P_1\right)\delta x_1$$

Extra work done at 2, as shown in Fig. 2.3 is

$$= P_2\,\delta x_2$$

Similarly extra work done at 3 etc. is $P_3\,\delta x_3$, etc.

Increase in strain energy = Total extra work done

$$\delta U = P_1\delta x_1 + P_2\delta x_2 + P_3\delta x_3 + \ldots \qquad \ldots(2.2)$$

If the loads $P_1 + \delta P_1$, P_2, P_3 etc. were applied gradually from Zero, the total strain energy would be

$$U + \delta U = \Sigma\frac{1}{2} \times \text{load} \times \text{deflection}$$

$$= \frac{1}{2}(P_1 + \delta P_1)(x_1 + \delta x_1) + \frac{1}{2}P_2(x_2 + \delta x_2)$$

$$+ \frac{1}{2}P_3(x_3 + \delta x_3) + \ldots$$

$$= \frac{1}{2}P_1x_1 + \frac{1}{2}P_1\delta x_1 + \frac{1}{2}\delta P_1 x_1 + \frac{1}{2}\delta P_1 \delta x_1$$

$$+\frac{1}{2}P_2x_2 + \frac{1}{2}P_2\delta x_2 + \frac{1}{2}P_3x_3 + \frac{1}{2}P_3\delta x_3 + \ldots \qquad \ldots(2.3)$$

Neglecting the product of small quantities and subtracting Eqn. (2.1) from (2.3), we get

$$\delta U = \delta P_1 x_1$$

or $$\frac{\delta U}{\delta P_1} = x_1$$

or In the limit $$\frac{\partial U}{\partial P_1} = x_1$$

$\Rightarrow$ The partial derivative of the strain energy with respect to P_1 give the deflection under P_1 in the direction of P_1. Similarly

$$\frac{\partial U}{\partial P_2} = x_2$$

and $$\frac{\partial U}{\partial P_3} = x_3 \text{ etc.}$$

or In general $$\frac{\partial U}{\partial P_i} = x_i \qquad \ldots(2.4a)$$

which is Castigliano's theorem.

Castigliano's theorem can also be applied to determine angular rotations under the action of bending moment or torque. In the same way, we can proceed to show that the angle of rotation

$$\theta_i = \frac{\partial U}{\partial M_i} \qquad \ldots(2.4b)$$

where M is the couple applied.

2.3.1 Deflection under Axial Load

Strain energy under axial load P,

$$U = \int \frac{1}{2}\frac{P^2dx}{AE}$$

$$\therefore \quad \delta = \frac{\partial U}{\partial P} = \int \frac{Pdx}{AE} \qquad \ldots(2.5)$$

2.3.2 Deflection under Bending

Strain energy under bending moment M,

$$U = \int \frac{M^2dx}{2EI}$$

$$\therefore \quad \delta = \frac{\partial U}{\partial P} = \frac{\partial U}{\partial M} \times \frac{\partial M}{\partial P}$$

$$= \int \frac{M}{EI}dx \times \frac{\partial M}{\partial P} = \int \frac{M}{EI}\frac{\partial M}{\partial P}dx \qquad \ldots(2.6)$$

2.3.3 Rotation under Torsion

Strain energy under torque T,

$$U = \int \frac{T^2dx}{2GJ}$$

$$\therefore \quad \theta = \frac{\partial U}{\partial T} = \int \frac{Tdx}{GJ} \qquad \ldots(2.7)$$

2.3.4 Rotation under Bending

$$U = \int \frac{M^2dx}{2EI}$$

$$\therefore \quad \phi = \frac{\partial U}{\partial M} = \int \frac{Mdx}{2EI} \qquad \ldots(2.8)$$

2.3.5 Deflection under Torsion

$$U = \int \frac{T^2dx}{2GJ}$$

$$\therefore \quad \delta = \frac{\partial U}{\partial P} = \frac{\partial U}{\partial T} \times \frac{\partial T}{\partial P}$$

$$= \int \frac{Tdx}{GJ}\frac{\partial T}{\partial P} = \int \frac{T}{GJ} \times \frac{\partial T}{\partial P}dx \qquad \ldots(2.9)$$

2.3.6 Deflection under Shear

Strain energy under shear force F,

$$U = \int \frac{F^2dx}{2AG}$$

$$\therefore \quad \delta = \frac{\partial U}{\partial F} = \int \frac{Fdx}{AG} \qquad \ldots(2.10)$$

Example 2.1 Using Castigliano's Theorem, obtain the deflection under a single concentrated load applied to a simply supported beam shown in Fig. 2.4

$$EI = 2 \text{ MN/m}^2$$

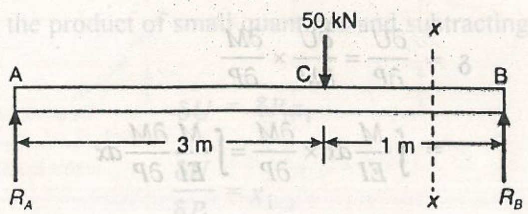


Fig. 2.4

Solution. Let the load at C be denoted by P kN.

$$R_A = \frac{P}{4} \text{ kN}$$

$$M_x = R_A x - P(x-3)$$

$$= \frac{Px}{4} - P(x-3)$$

$$\frac{\partial M}{\partial P} = \frac{x}{4} - (x-3)$$

$$\delta = \int \frac{M}{EI}\frac{\partial M}{\partial P} dx$$

$$= \frac{1}{EI}\int_0^3 \frac{Px}{4} \times \frac{x}{4} dx + \frac{1}{EI}\int_3^4 \left\{\frac{Px}{4} - P(x-3)\right\}\left\{\frac{x}{4} - (x-3)\right\} dx$$

$$= \frac{P}{16EI}\left|\frac{x^3}{3}\right|_0^3 + \frac{9P}{16EI}\left|16x + \frac{x^3}{3} - 4x^2\right|_3^4$$

$$= \frac{9P}{16EI} + \frac{9P}{16EI}[16 + 12.33 - 28]$$

$$= \frac{3P}{4EI}$$

$$\delta = \frac{3 \times 50 \times 10^3}{4 \times 2 \times 10^6} = 1.875 \times 10^{-2} \text{ m} = 18.75 \text{ mm}$$

Example 2.2 Using Castigliano's theorem, obtain the deflection at the centre of a beam carrying u.d.l. of 20 kN/m over the whole span. The beam is simply supported over a span of 3 m and $EI = 2.5$ MN-m^2.

Solution. Let P be the dummy load at the mid span. Then taking moments about the right hand support.

$$R_A \times 3 = 20 \times 3 \times 1.5 + P \times 1.5 = 90 + 1.5\,P$$

$$\Rightarrow \quad R_A = (30 + 0.5\,P) \text{ kN}$$

$$M_x = (30 + 0.5\,P)x - 10x^2 - P(x - 1.5)$$

$$\frac{\partial M_x}{\partial P} = 0.5x - (x - 1.5)$$

$$\delta = \int \frac{M}{EI}\frac{\partial M}{\partial P} dx = \frac{1}{EI}\int_0^{1.5}\left\{(30 + 0.5P)x - 10x^2\right\}0.5x\,dx$$

$$+ \frac{1}{EI}\int_{1.5}^3 \left\{(30 + 0.5P)x - 10x^2 - P(x - 1.5)\right\}\left\{0.5x - (x - 1.5)\right\} dx$$

$$= \frac{1}{EI}\left[\left|(15 + 0.25P)\frac{x^3}{3} - \frac{5x^4}{4}\right|_0^{1.5} + \frac{1}{EI}\left|(45 - 1.5P)\frac{x^2}{2} + (0.25P - 30)\frac{x^3}{3} + \frac{5x^4}{4} + 2.25Px\right|_{1.5}^3\right]$$

Putting $P = 0$, we get

$$\delta = \frac{1}{EI}\left[5 \times 1.5^3 - 1.25 \times 1.5^4\right] + \frac{1}{EI}\left[22.5 \times 6.75 - 10 \times 23.625 + 1.25 \times 75.9375\right]$$

$$= \frac{21.0938}{EI} = \frac{21.0938 \times 10^3}{2.5 \times 10^6} = 8.4375 \times 10^{-3} \text{ m}$$

= 8.4375 mm **Ans**

Example 2.3 A simply supported beam with overhang is loaded as shown in Fig. 2.5. Using the Castigliano's theorem, find the vertical deflection of point C.

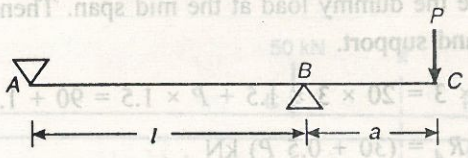


Fig. 2.5

Solution. Following is the bending moment diagram for the beam (Fig.2.6).

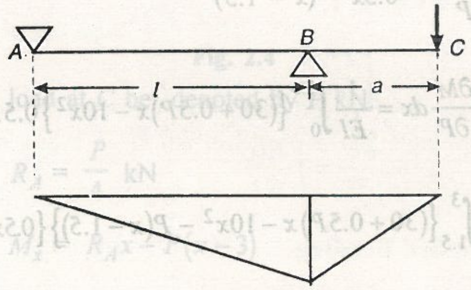


Fig. 2.6

Between A and B, a general expression for B.M. at x distance to the right of A is

$$M_x = -\frac{Pax}{l}$$

Between B and C, the B.M. at any distance x to the left of C is

$$M_x = -Px$$

The strain energy of bending in the beam becomes

$$U = \int_0^l \frac{M_x^2 dx}{2EI}$$

$$\Rightarrow \qquad U = \int_0^l \frac{P^2a^2x^2}{2EIl^2}dx + \int_0^a \frac{P^2x^2}{2EI}dx$$

$$= \frac{P^2a^2}{6EI}(l+a)$$

$$\therefore \qquad \delta = \frac{\partial U}{\partial P} = \frac{Pa^2}{3EI}(l+a) \quad \textbf{Ans.}$$

Example 2.4 The axis of a cantilever ring, built in at B and loaded at the free end A, forms a horizontal quarter circular arc of radius R. (Fig. 2.7). Find the vertical deflection δ, of the free end A, assuming the ring to have a circular cross section, the diameter of which is small compared with the radius R of its centre line.

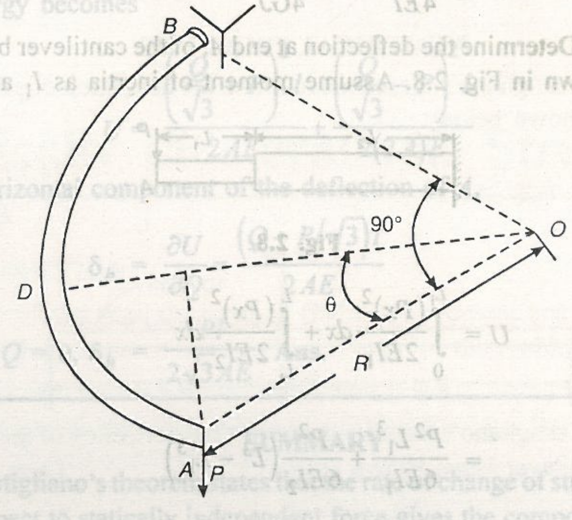


Fig. 2.7

Solution. Such a curved cantilever is subjected to torsion as well as bending. At any cross section D, defined by the angle θ, the bending moment is

$$M_\theta = PR\sin\theta \qquad \ldots(1)$$

The twisting moment on the same cross-section is

$$T_\theta = PR(1-\cos\theta) \qquad \ldots(2)$$

The total strain energy due to the combined bending and torsion of the ring is

$$U = \int_0^{\pi/2} \frac{M_\theta^2}{2EI} Rd\theta + \int_0^{\pi/2} \frac{T_\theta^2}{2GJ} Rd\theta$$

$$\therefore \qquad \delta = \frac{\partial U}{\partial P} = \int_0^{\pi/2} \frac{M_\theta\left(\frac{\partial M_\theta}{\partial P}\right)Rd\theta}{2EI} + \int_0^{\pi/2} \frac{T_\theta\left(\frac{\partial T_\theta}{\partial P}\right)Rd\theta}{2GJ} \qquad \ldots(3)$$

From Eqns. (1) and (2)

$$\frac{\partial M_\theta}{\partial P} = R\sin\theta$$

$$\frac{\partial T_\theta}{\partial P} = R(1-\cos\theta)$$

Putting these values in Eqn. (3)

$$\delta = \frac{\pi PR^3}{4EI} + \frac{(3\pi - 8)PR^3}{4GJ}$$

Example 2.5 Determine the deflection at end *A*, of the cantilever beam of total length *L*, shown in Fig. 2.8. Assume moment of inertia as I_1 and I_2.

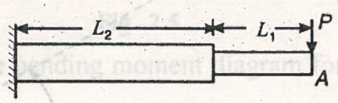


Fig. 2.8

Solution.

$$U = \int_0^{L_1} \frac{(Px)^2}{2EI_1}dx + \int_{L_1}^{L} \frac{(Px)^2}{2EI_2}dx$$

$$= \frac{P^2 L_1^3}{6EI_1} + \frac{P^2}{6EI_2}\left(L^3 - L_1^3\right)$$

$$\delta = \frac{\partial U}{\partial P} = \frac{PL_1^3}{3EI_1} + \frac{P}{3EI_2}\left(L^3 - L_1^3\right)$$

Example 2.6 A simple truss composed of two bars each of length *l* carries a vertical load *P* at joint *A* as shown in Fig. 2.9. Find the horizontal and vertical components of the total deflection δ of point *A*. The bars are of the same material, *AB* having a cross sectional area *A* and *AC* has a cross sectional area $A_1 = 2A$.

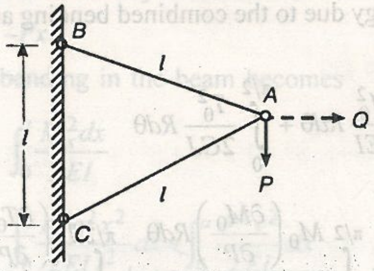


Fig. 2.9

Solution. From statics, the tensile force in the bar *AB* and the compressive force in the bar *AC* are equal to *P*. Hence the strain energy of system

$$U = \frac{P^2 l}{2AE} + \frac{P^2 l}{2A_1 E} = \frac{3P^2 l}{4AE}$$

Vertical component of the deflection of *A*,

$$\delta_v = \frac{\partial U}{\partial P} = \frac{3Pl}{2AE} \quad \textbf{Ans.}$$

To find the horizontal component of the deflection of joint we introduce a fictitious horizontal force *Q* at *A*, with this force acting in addition to *P*, the strain energy becomes

$$U = \frac{\left(\frac{Q}{\sqrt{3}} + P\right)^2 l}{2AE} + \frac{\left(\frac{Q}{\sqrt{3}} - P\right)^2 l}{2(2A)E}$$

∴ Horizontal component of the deflection of *A*,

$$\delta_h = \frac{\partial U}{\partial Q} = \frac{\left(Q + P/\sqrt{3}\right)l}{2AE}$$

$$\text{Putting } Q = 0,\ \delta_h = \frac{Pl}{2\sqrt{3}AE} \quad \textbf{Ans.}$$

SUMMARY

1. Castigliano's theorem states that the rate of change of strain energy with respect to statically independent force gives the component deflection of this force in the direction of the force.
2. Castigliano's theorem is extremely useful in determining the displacements of structures.
3. Castigliano's theorem can also be applied to determine angular rotations under the action of bending moment or torque.
4. Castigliano's theorem gives the displacement of point in the direction of the external forces, where they are acting.
5. This theorem states that if a body is subjected to a number of loads, the partial derivative of the total strain energy with respect to any load gives the deflection in the direction of that load.
6. In general $\frac{\partial U}{\partial P_i} = x_i$ or $\frac{\partial U}{\partial M_i} = \theta_i$ is Castigliano's theorem.
7. While applying Castigliano's theorem, treat all the loads as 'Variables' initially, carry out the partial differentiation and integration, putting in numerical values at the final stage.
8. If the deflection is to be found (by using Castigliano's theorem) at a point where, or in a direction in which, there is no load, a load may be put in where required and given a value zero in the final reckoning.

   i.e. $x = \left(\frac{\partial U}{\partial W}\right)_{W=0}$

REVIEW QUESTIONS

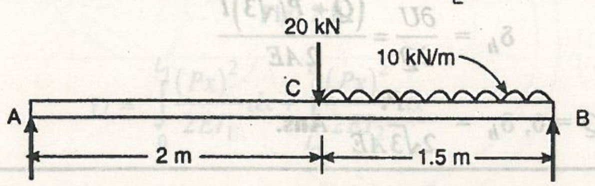
Write short notes on the following :

1. Castigliano's theorem (only statement)
2. Areas of application of Castigliano's theorem
3. Derive the Castigliano's theorem.

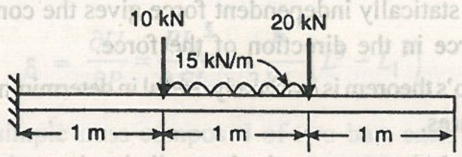
NUMERICAL PROBLEMS

1. Using Castigliano's theorem determine the deflection of point C of the beam shown below

$$\left[\textbf{Ans.}\ \frac{24.772\times10^3}{EI}\ \text{m}\right]$$



2. Using Castigliano's theorem determine the deflection of the free end of the cantilever beam shown below. **[Ans.** 37.4 mm]



3. A simply supported beam of span 2 m is carrying two loads of 500 kN and 1000 kN at 1 m and 1.5 m respectively from the left hand support. Using Castigliano's theorem, calculate the deflection at the centre. Take E = 210 GPa, $I = 70 \times 10^{-6}$ m^4. **[Ans.** 12 mm]
4. A simply supported beam of span 3 m carries a u.d.l. of 200 kN/m. Calculate the deflection at the centre by using Castigliano's theorem. E = 200 GPa, $I = 10^{-4}$ m^4 **[Ans.** 10.5 mm]
5. A rectangular section ($b \times d$) cantilever of length l carries u.d.l. spread from free end to the mid section of the cantilever. Find the deflection due to shear at the free end with the help of Castigliano's theorem.

$$\left[\textbf{Ans.}\ \frac{qwl^2}{20Gbd}\right]$$

3

Statically Indeterminate Structures

3.1 INTRODUCTION

Previously, we have dealt the cases of finding out stresses and strains where simple equations of statics were sufficient to solve the problems. But some times the simple equations are not sufficient to solve such problems. Such problems are called statically indeterminate problems and the structures are called *statically indeterminate structures*.

For solving statically indeterminate problems, the deformation characteristics of the structure are also taken into account alongwith the statical equilibrium equations. Such equations which contain the deformation characteristics, are called *compatibility equations*. Thus, in this chapter we will come across such problems, in which force in the member is not found just by using equations of statics. First we will consider the nature of deformation (compatibility equations) to get additional equations and finally we will solve the set of equations of statics and compatibility.

Before we carry on further discussion and discuss the various types of statically indeterminate structures, let us go through a few sample problems.

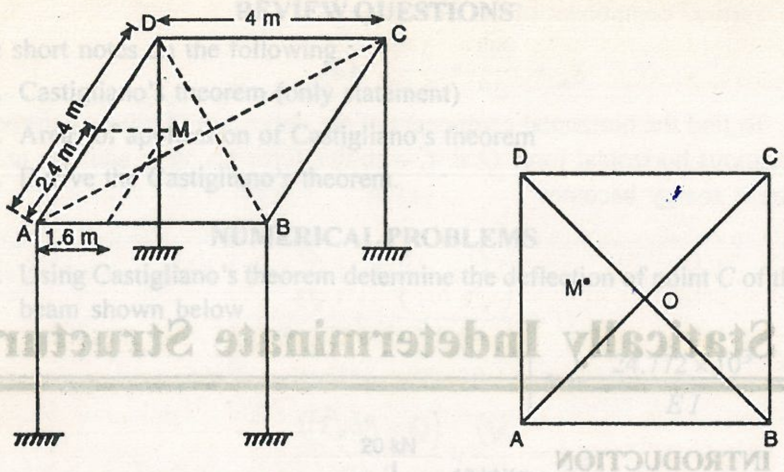
Sample Problem 3.1 Four identical pillars located at points A, B, C and D are of height 4 m and are supporting a rigid platform as shown in Fig. 3.1. What are the forces introduced on each of the columns due to a load of 20 kN placed at point M on the rigid platform ?

Solution. Let the forces introduced in the pillars be P_A, P_B, P_C and P_D. The three equations of statics can be written as

$$\Sigma \text{ Vertical forces} = 0$$

$$P_A + P_B + P_C + P_D = 20 \quad \ldots(1)$$

$$\Sigma M_{AB} = 0$$



(*a*) (*b*)

Fig. 3.1

$$(P_D + P_C)4 = 20 \times 2.4$$

$$\Rightarrow \quad P_D + P_C = 12 \qquad \ldots(2)$$

$$\Sigma M_{AD} = 0$$

$$(P_C + P_B)4 = 20 \times 1.6$$

$$\text{or} \quad P_C + P_B = 8$$

Let $\Delta_A, \Delta_B, \Delta_C$ and Δ_D be the deformations of the pillars *A*, *B*, *C* and *D* respectively. As the platform is rigid, diagonal *AC* and *AD* remain straight lines even after the load is applied. Hence, the deflection of central point *O* is given by $\frac{\Delta_A + \Delta_C}{2}$ and also by $\frac{\Delta_B + \Delta_D}{2}$

$$\Rightarrow \quad \Delta_A + \Delta_C = \Delta_B + \Delta_D$$

$$\Rightarrow \quad \frac{P_A L}{AE} + \frac{P_C L}{AE} = \frac{P_B L}{AE} + \frac{P_D L}{AE}$$

Since pillars are identical in length, cross sectional area and material property (Young's modulus *E*), we get

$$P_A + P_C = P_B + P_D \qquad \ldots(4)$$

The above four equations can be solved as given below to get four unknowns P_A, P_B, P_C and P_D. From equation (1) and (4), we get

$$P_B + P_D = 10 \qquad \ldots(5)$$

Subtracting equation (3) from (2), we get

$$P_D - P_B = 4 \qquad \ldots(6)$$

Adding equation (5) and (6), we get

PD = 7 kN

Substituting it in (3) we get, $P_B = 3$ kN

Substituting P_B, P_C and P_D in (1), we get

$$P_A = 5 \text{ kN}$$

Thus the forces developed in the pillars are

$P_A = 5$ kN, $P_B = 3$ kN, $P_C = 5$ kN, $P_D = 7$ kN **Ans.**

Sample Problem 3.2 **Four identical wires support a rigid bar as shown in Fig. 3.2. If the stress is not to exceed 150 N/mm^2, find the minimum required diameter of the wires to support a load of 40 kN.**

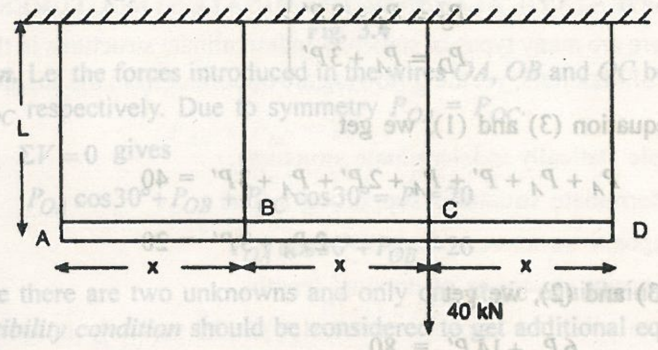


Fig. 3.2

Solution. **Let P_A, P_B, P_C and P_D be the forces developed in the wires *A*, *B*, *C* and *D* respectively. The *equations of statics* are,**

$$\Sigma V = 0 \quad \Rightarrow \quad P_A + P_B + P_C + P_D = 40 \text{ kN} \qquad \ldots(1)$$

$$\Sigma M_A = 0 \quad \Rightarrow \quad P_B x + P_C 2x + P_D 3x = 40.2x$$

$$\Rightarrow \quad P_B + 2P_C + 3P_D = 80 \qquad \ldots(2)$$

As the two equations of the statics are not enough to find the four unknowns, the *deformation characteristics* are to be considered to get compatibility equations. Since the bar is rigid, deformed shape will be as shown in Fig. (3.3). Let Δ be increase in elongation of wire *B* over that of wire *A*. Then

$$\Delta_B = \Delta_A + \Delta$$

$$\Delta_C = \Delta_A + 2\Delta$$

$$\Delta_D = \Delta_A + 3\Delta$$

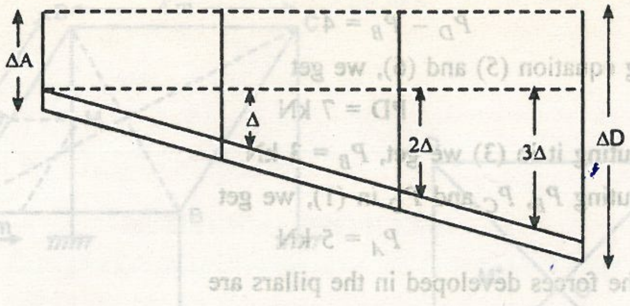


Fig. 3.3

If P' is the force required for elongation Δ of the wires, then

$$\left.\begin{aligned} P_B &= P_A + P' \\ P_C &= P_A + 2P' \\ P_D &= P_A + 3P' \end{aligned}\right\} \qquad ...(3)$$

From equation (3) and (1), we get

$$P_A + P_A + P' + P_A + 2P' + P_A + 3P' = 40$$

$$\Rightarrow \qquad 2P_A + 3P' = 20 \qquad ...(4)$$

From (3) and (2), we get

$$6P_A + 14P' = 80 \qquad ...(5)$$

Subtracting 3 times equation (4) from equation (5)

$$P' = 4 \text{ kN}$$

From equation (4), $P_A = 4$ kN

$$\therefore \qquad P_B = P_A + P' = 8 \text{ kN}$$

$$P_C = P_A + 2P' = 12 \text{ kN}$$

$$P_D = P_A + 3P' = 16 \text{ kN} \textbf{ Ans.}$$

Maximum load = 16 kN, Permissible stress = 150 N/mm²

$\therefore$ If d is the diameter required then

$$\frac{\pi}{4} d^2 \times 150 = 16 \times 10^3$$

$$\Rightarrow \qquad d = 11.6538 \text{ mm}$$

Hence, required minimum diameter = 11.6538 mm **Ans.**

Sample Problem 3.3 Three identical wires support a load of 20 kN as shown in Fig. 3.4(*a*). Determine the force in each wire and the vertical displacement of the load, if diameter of each wire is 6 mm and $E = 2 \times 10^5$ N/mm².

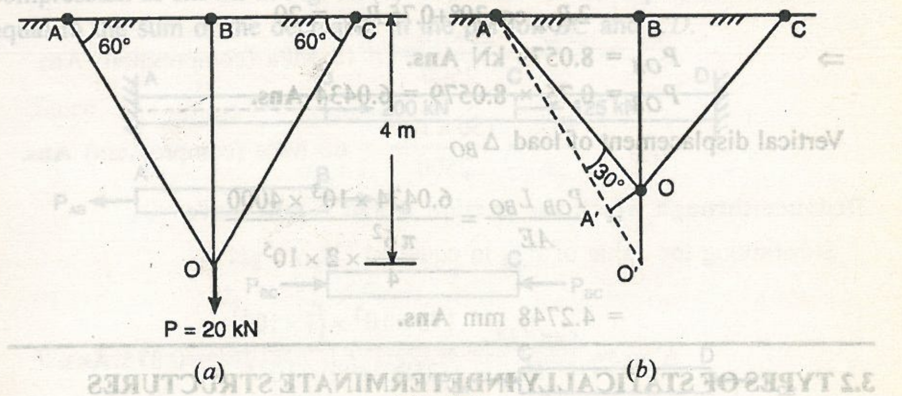


Fig. 3.4

Solution. Let the forces introduced in the wires *OA*, *OB* and *OC* be P_{OA}, P_{OB} and P_{OC} respectively. Due to symmetry $P_{OA} = P_{OC}$.

$\Sigma V = 0$ gives

$$P_{OA} \cos 30° + P_{OB} + P_{OC} \cos 30° = P = 20$$

or

$$P_{OA} \cos 30° + P_{OB} = 20 \qquad ...(1)$$

Since there are two unknowns and only one static equilibrium equation, *compatibility condition* should be considered to get additional equation.

Fig. 3.4(*b*) shows deformation characters of this structure. Let O' be the final position of load. Join AO'. Draw arc from *A* with a radius of L_{AO} cutting AO' at A', then OA' my be taken as at right angle to AO' and $A'O'$ as extension of wire AO. OO' is extension of BO.

From Fig. 3.4(*b*)

$$OO' = \frac{A'O}{\cos 30°}$$

$$\Rightarrow \qquad \text{Extension of wire } BO = \frac{\text{Extension of wire } AO}{\cos 30°}$$

$$\Rightarrow \qquad \frac{P_{OB} L_{BO}}{AE} = \frac{P_{OA} L_{AO}}{AE \cos 30°}$$

$$\Rightarrow \qquad P_{OB} L_{BO} = \frac{P_{OA} L_{AO}}{\cos 30°}$$

$$L_{AO} = \frac{L_{BO}}{\cos 30°}$$

$$\therefore \quad P_{OB} = \frac{P_{OA}}{\cos^2 30°} = 0.75\ P_{OA} \quad ...(2)$$

Substituting it in equaiton (1), we get

$$2P_{OA}\cos 30° + 0.75 P_{OA} = 20$$

$$\Rightarrow \quad P_{OA} = 8.0579 \text{ kN } \textbf{Ans.}$$

$$P_{OB} = 0.75 \times 8.0579 = 6.0434 \textbf{ Ans.}$$

Vertical displacement of load Δ_{BO}

$$= \frac{P_{OB}\,L_{BO}}{AE} = \frac{6.0434 \times 10^3 \times 4000}{\frac{\pi 6^2}{4} \times 2 \times 10^5}$$

$$= 4.2748 \text{ mm } \textbf{Ans.}$$

3.2 TYPES OF STATICALLY INDETERMINATE STRUCTURES

Though, there are many types of statically indeterminate structures in the field of Strength of Materials, yet the following are important from the subject point of view :

1. Simple statically indeterminate structures.
2. Indeterminate structures supporting a load.
3. Composite structures of equal lengths.
4. Composite structures of unequal lengths.

In order to solve the above mentioned types of statically indeterminate structures, we have to use different types of compatible equations.

3.3 STRESSES IN SIMPLE STATICALLY INDETERMINE STRUCTURES

The structures in which the stresses can be obtained by forming two or more equations are called as simple statically indeterminate structures. The stresses in such structures may be found out with the help of two or three compatible equations.

Example 3.1 An aluminium bar 3 m long and 2500 mm² in cross section is rigidly fixed at *A* and *D* as shown in Fig. 3.5(*a*).

Fig. 3.5(*a*)

Determine the loads shared and stresses in each portion and the distances through which the portion *B* and *C* will move. Take *E* for aluminium as 80 GPa.

Solution. Let loads shared by the different portions are P_{AB}, P_{BC} and P_{CD}. Since the bar is rigidly fixed at *A* and *D*, therefore the portion *AB* will be subjected to tension, while the portions *BC* and *CD* will be subjected to compression as shown in Fig. 3.5(*b*). The increase in the portion *AB* will be equal to the sum of the decreases in the portion *BC* and *CD*.

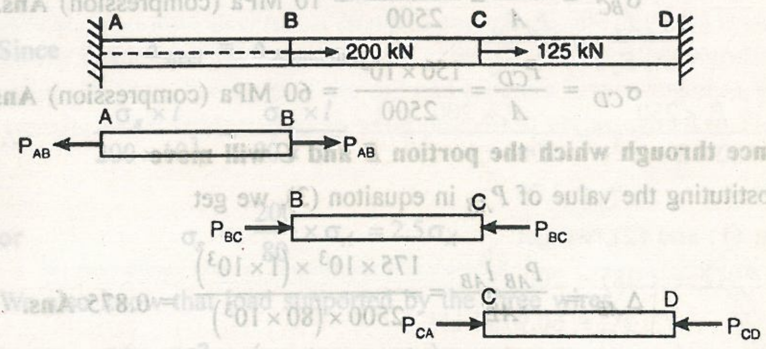


Fig. 3.5(*b*)

From the geometry of the bar, we find that

$$P_{AB} + P_{BC} = 200$$

or $$P_{AB} = 200 - P_{BC} \quad ...(1)$$

and $$P_{CD} - P_{BC} = 125$$

or $$P_{CD} = 125 + P_{BC} \quad ...(2)$$

$$\Delta_{AB} = \frac{P_{AB}\,l_{BC}}{AE} = \frac{P_{AB}\left(1 \times 10^3\right)}{AE} \quad ...(3)$$

$$\Delta_{BC} = \frac{P_{BC}\,l_{BC}}{AE} = \frac{P_{BC}\left(1 \times 10^3\right)}{AE} \quad ...(4)$$

$$\Delta_{CD} = \frac{P_{CD}\,l_{CD}}{AE} = \frac{P_{CD}\left(1 \times 10^3\right)}{AE} \quad ...(5)$$

Since $$\Delta_{AB} = \Delta_{BC} + \Delta_{CD}$$

$$\therefore \quad \frac{P_{AB} \times 10^3}{AE} = \frac{P_{BC} \times 10^3}{AE} + \frac{P_{CD} \times 10^3}{AE}$$

$$\Rightarrow \quad P_{AB} = P_{BC} + P_{CD}$$

Putting the values P_{AB} and P_{CD} from equaiton (1) and (2) in it

$$(200 - P_{BC}) = P_{BC} + (125 + P_{BC})$$

$$\Rightarrow \quad P_{BC} = 25 \text{ kN}$$

$$\therefore \quad P_{AB} = 200 - 25 = 175 \text{ kN } \textbf{Ans.}$$

$$P_{CD} = 125 + 25 = 150 \text{ kN } \textbf{Ans.}$$

Stresses in each portion

$$\sigma_{AB} = \frac{P_{AB}}{A} = \frac{175 \times 10^3}{2500} = 70 \text{ MPa (tension) } \textbf{Ans.}$$

$$\sigma_{BC} = \frac{P_{BC}}{A} = \frac{25 \times 10^3}{2500} = 10 \text{ MPa (compression) } \textbf{Ans.}$$

$$\sigma_{CD} = \frac{P_{CD}}{A} = \frac{150 \times 10^3}{2500} = 60 \text{ MPa (compression) } \textbf{Ans.}$$

Distance through which the portion *B* and *C* will move

Substituting the value of P_{AB} in equaiton (3), we get

$$\Delta_{AB} = \frac{P_{AB}\, l_{AB}}{AE} = \frac{175 \times 10^3 \times (1 \times 10^3)}{2500 \times (80 \times 10^3)} = 0.875 \textbf{ Ans.}$$

Substituting the value of P_{CD} in equation (4), we get

$$\Delta_{CD} = \frac{P_{CD}\, l_{CD}}{AE} = \frac{(150 \times 10^3) \times (1 \times 10^3)}{2500 \times (80 \times 10^3)} = 0.75 \text{ mm } \textbf{Ans.}$$

3.4 STRESSES IN INDETERMINATE STRUCTURES SUPPORTING A LOAD

Sometimes, we find a set of two or more members supporting a load. In such cases, the deformation of all the members will be the same. If the members are of different cross-sections or have different modulus of elasticity, then the stresses developed in all the members will be different.

Example 3.2 A block shown in Fig. 3.6 weighing 30 kN is supported by three wires. The outer two wires are of steel and have an area of 100 mm² each, whereas the middle wire is of aluminium and has an area of 200 mm². If the elastic modulii of steel and aluminium are 200 GPa and 80 GPa respectively, then calculate the stresses in the aluminium and steel wires.

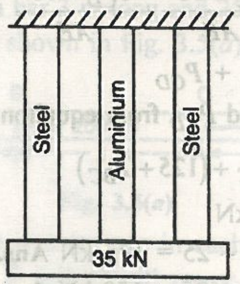


Fig. 3.6

Solution.

$$\Delta_{steel} = \frac{\Delta_{steel} \times l_{steel}}{E_{steel}} = \frac{\sigma_s \times l}{200 \times 10^3} \quad \ldots(1)$$

$$\sigma_{Aluminium} = \frac{\sigma_{Aluminium} \times l_{Aluminium}}{E_{Aluminium}} = \frac{\sigma_A \times l}{80 \times 10^3} \quad \ldots(2)$$

Since $\Delta_{steel} = \Delta_{Aluminium}$

$$\therefore \quad \frac{\sigma_s \times l}{200 \times 10^3} = \frac{\sigma_A \times l}{80 \times 10^3}$$

or $$\sigma_s = \frac{200}{80} \times \sigma_A = 2.5\sigma_A$$

We also know that load supported by the three wires

$$30 \times 10^3 = (\sigma_s A_s + \sigma_A A_A)$$

$$= (2.5\sigma_A \times 200) + (\sigma_A \times 200) = 700\sigma_A$$

$$\sigma_A = 50 \text{ MPa } \textbf{Ans.}$$

$$\sigma_s = 2.5\sigma_A = 2.5 \times 50 = 125 \text{ MPa } \textbf{Ans.}$$

Example 3.3 Three pillars, two of aluminium and one of steel, support a rigid platform of 200 kN as shown in Fig. 3.7. If area of each aluminium pillar is 1000 mm² and that of steel pillar is 800 mm², find the stresses developed in each pillar. Take $E_a = 1 \times 10^5$ N/mm² and $E_s = 2 \times 10^5$ N/mm². What additional load *P* it can take if working stresses are 65 N/mm² in aluminium and 150 N/mm² in steel ?

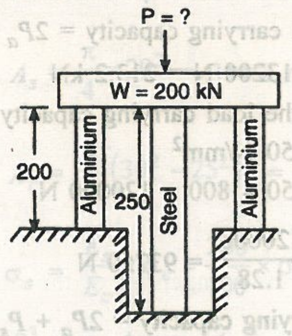


Fig. 3.7

Solution. (*i*) Due to 200 kN load only :

Let P_a be the force in each of aluminium pillars and P_s be the force in steel pillar. Then

Σ force in vertical direction = 0

$$\Rightarrow \quad P_a + P_s + P_a = 200$$

$$\text{or} \quad 2P_a + P_s = 200 \quad \text{...(1)}$$

Compatibility condition gives, $\Delta_s = \Delta_a$

$$\frac{P_s L_s}{A_s E_s} = \frac{P_a L_a}{A_a E_a}$$

$$\frac{P_s \times 250}{800 \times 2 \times 10^5} = \frac{P_s \times 200}{1000 \times 1 \times 10^5}$$

$$P_s = 1.28 \text{ Pa} \quad \text{...(2)}$$

From (1) and (2), we get

$$P_a(2 + 1.28) = 200$$

$$P_a = 60.9756 \text{ kN}$$

$$\therefore \quad P_s = 78.0488 \text{ kN}$$

Stresses developed are :

$$\sigma_{\text{aluminium}} = \frac{60.9756 \times 1000}{1000} = 60.9756 \text{ N/mm}^2$$

$$\sigma_{\text{steel}} = \frac{78.0488 \times 1000}{800} = 97.5610 \text{ N/mm}^2$$

(*ii*) Additional load carrying capacity :

(*a*) If p_a governs the load carrying capacity then $p_a = 65 \text{ N/mm}^2$

$$P_a = 65 \times 1000 = 65000 \text{ N}$$

$$P_s = 1.28 \times 65000 = 83200 \text{ N}$$

$$\therefore \quad \text{Total load carrying capacity} = 2P_a + P_s$$

$$= 213200 \text{ N} = 213.2 \text{ kN}$$

(*b*) If p_s governs the load carrying capacity then

$$p_s = 150 \text{ N/mm}^2$$

$$P_s = 150 \times 800 = 120000 \text{ N}$$

$$P_a = \frac{120000}{1.28} = 93750 \text{ N}$$

Total load carrying capacity $= 2P_a + P_s = 307500$ N

$= 307.5$ kN

Hence, actual load carrying capacity is 213.2 kN, which means, additional load $P = 213.2 - 200$

$= 13.2$ kN

3.5 STRESSES IN COMPOSITE STRUCTURES OF EQUAL LENGTHS

There are many types of such structures, yet a rod passing axially through a pipe is an important structure from the subject point of view.

Exercise 3.4 A mild steel rod of 20 mm diameter and 300 mm long is enclosed centrally inside a hollow copper tube of external diameter 30 mm and internal diameter 25 mm. The ends of the rod and tube are brazed together, and the composite bar is subjected to an axial pull of 40 kN as shown in Fig. 3.8.

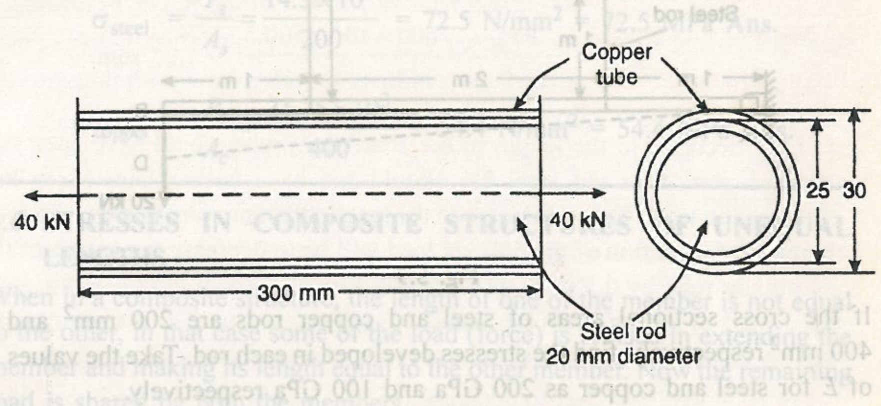


Fig. 3.8

If E for steel and copper is 200 GPa and 100 GPa respectively, find the stresses developed in the rod and the tube.

Solution. Let σ_s = stress developed in the steel rod

σ_c = stress developed in the copper tube

$$\text{Area of steel rod} \quad A_s = \frac{\pi}{4} \times 20^2 = 314.2 \text{ mm}^2$$

$$\text{Area of copper tube} \quad A_c = \frac{\pi}{4}\left[\left(30^2 - 25^2\right)\right] = 216 \text{ mm}^2$$

$$\sigma_s = \frac{E_s}{E_c} \times \sigma_c = \frac{200}{100} \times \sigma_c = 2\sigma_c$$

$$\text{Total load } (P), \; 40 \times 10^3 = \sigma_s A_s + \sigma_c A_c$$

$$= (2\sigma_c \times 314.2) + (\sigma_c \times 216) = 844.4\,\sigma_c$$

$$\therefore \quad \sigma_c = 47.4 \text{ N/mm}^2 = 47.4 \text{ MPa } \textbf{Ans.}$$

$$\text{and} \quad \sigma_s = 2\sigma_c = 2 \times 47.4 = 94.8 \text{ MPa } \textbf{Ans.}$$

Example 3.5 A rigid bar *AB* is hinged at *A* and supported by a copper rod 2 m long and steel rod 1 m long. The bar carries a load of 20 kN at *D* as shown in Fig. 3.9.

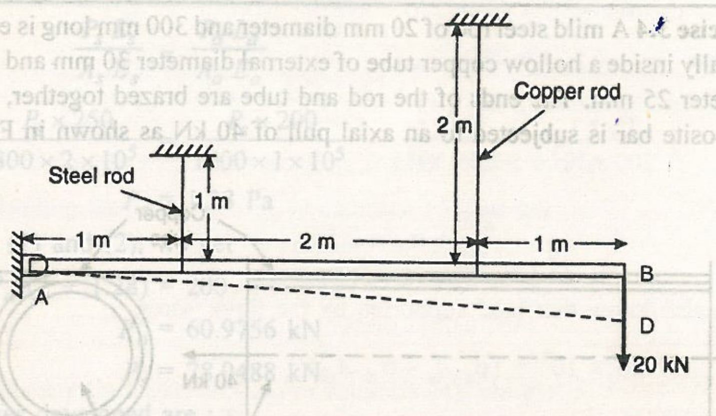


Fig. 3.9

If the cross sectional areas of steel and copper rods are 200 mm^2 and 400 mm^2 respectively, find the stresses developed in each rod. Take the values of *E* for steel and copper as 200 GPa and 100 GPa respectively.

Solution. Let P_s = load shared by the steel rod

P_c = load shared by the copper rod

Taking moments of the loads about *A* and equating

$$(P_s \times 1) + (P_c \times 3) = 20 \times 4$$

$$\Rightarrow \quad P_s + 3P_c = 80 \qquad ...(1)$$

$$\Delta_{steel} = \frac{P_s l_s}{A_s E_s} = \frac{P_s \times (1 \times 10^3)}{200 \times (200 \times 10^3)} = 0.025 \times 10^{-3} P_s \qquad ...(2)$$

$$\Delta_{copper} = \frac{P_c l_c}{A_c E_c} = \frac{P_c \times (2 \times 10^3)}{400 \times (100 \times 10^3)} = 0.05 \times 10^{-3} P_c \qquad ...(3)$$

From the geometry of the elongations of the steel rod and copper rod,

$$\frac{\Delta_{steel}}{1} = \frac{\Delta_{copper}}{3}$$

$$\Rightarrow \quad \Delta_{copper} = 3\Delta_{steel}$$

Putting the values of Δ_{steel} and Δ_{copper} from Eqns. (2) and (3) in this equation

$$0.05 \times 10^3 P_c = 3 \times 0.025 \times 10^3 P_s$$

or $\quad P_c = 1.5 P_s$

Putting the value of P_c in equation (1)

$$P_s + 3 \times (1.5 P_s) = 80$$

or $\quad P_s = 14.5 \times 10^3$ N

and $\quad P_c = 21.75 \times 10^3$ N

$$\sigma_{steel} = \frac{P_s}{A_s} = \frac{14.5 \times 10^3}{200} = 72.5 \text{ N/mm}^2 = 72.5 \text{ MPa } \textbf{Ans.}$$

$$\sigma_{copper} = \frac{P_c}{A_c} = \frac{21.75 \times 10^3}{400} = 54.4 \text{ N/mm}^2 = 54.4 \text{ MPa } \textbf{Ans.}$$

3.6 STRESSES IN COMPOSITE STRUCTURES OF UNEQUAL LENGTHS

When in a composite structure, the length of one of the member is not equal to the other, in that case some of the load (force) is utilised in extending the member and making its length equal to the other member. Now the remaining load is shared by both the members.

Example 3.6 A solid steel bar 500 mm long and 50 mm diameter is placed inside an aluminium tube of 75 mm inside diameter and 100 mm outside diameter.

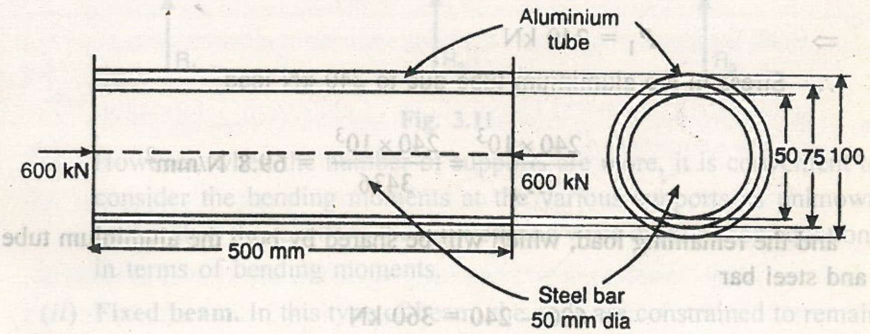


Fig. 3.10

The aluminium tube is 0.5 mm longer than the steel bar. An axial load of 600 kN is applied to the bar and cylinder through rigid plates as shown in Fig. 3.10. Find the stresses developed in the steel bar and aluminium tube. Assume *E* for steel as 200 GPa and *E* for aluminium is 70 GPa.

Solution. Area of steel bar $A_s = \frac{\pi}{4} \times 50^2 = 1964$ mm^2

Area of aluminium tube $A_A = \frac{\pi}{4}\left[D_A^2 - d_A^2\right]$

$$= \frac{\pi}{4}\left[100^2 - 75^2\right] = 3436 \text{ mm}^2$$

Since the aluminium tube is longer than the steel bar by 0.5 mm, therefore the load will first come upon the tube. Therefore, decrease in the length of the aluminium tube due to load,

$$\Delta_{Al} = \frac{P\,l_{Al}}{A_A E_{Al}} = \frac{600 \times 10^3 \times 500.5}{3436 \times (70 \times 10^3)} = 1.25 \text{ mm}$$

If the decrease in the length of the aluminium tube would have been less than 0.5 mm, then the steel bar should not have been subjected to any compressive load. As the decrease in the length of aluminium tube is 1.25 mm, therefore, first action of the 600 kN load will be to decrease the length of the aluminium tube by 0.5 mm, till its length becomes equal to that of the steel bar. A part of the load will be required for this decrease. The remaining load will be shared by both the aluminium tube and steel bar.

Let P_1 = load required to decrease 0.5 mm length of the aluminium tube.

We know that decrease in length

$$0.5 = \frac{P_l\, l_A}{A_A E_A} = \frac{P_l \times 500.5}{3436 \times (70 \times 10^3)} = 2.08 \times 10^{-6} P_1$$

$$\Rightarrow \quad P_1 = 240 \text{ kN}$$

$\therefore$ Stress in the aluminium tube due to 240 kN load

$$= \frac{240 \times 10^3}{A_A} = \frac{240 \times 10^3}{3436} = 69.8 \text{ N/mm}^2$$

and the remaining load, which will be shared by both the aluminium tube and steel bar

$$= 600 - 240 = 360 \text{ kN}$$

Let σ_A = stress developed in the aluminium tube due to 360 kN load

and σ_s = stress developed in the steel bar due to 360 kN load

Stress in steel $\sigma_s = \frac{E_s}{E_A} \times \sigma_A = \frac{200}{70} \times \sigma_A = 2.86\,\sigma_A$

and the load shared by both the aluminium tube and steel bar,

$$360 \times 10^3 = \sigma_s A_s + \sigma_A A_A$$

$$= (2.86\sigma_A \times 1964) + (\sigma_A \times 3436) = 9053\,\sigma_A$$

$$\therefore \quad \sigma_A = 39.8 \text{ N/mm}^2$$

and $\sigma_s = 2.86\sigma_A = 2.86 \times 39.8 = 113.8 \text{ N/mm}^2$

= 113.8 MPa **Ans.**

3.7 STATICALLY INDETEMINATE BEAMS

Beams for which the equations of equilibrium alone are insufficient to determine the internal forces are called statically indeterminate.

To determine the reactions at the supports, we will have to use the equations of displacements of deformations, additional to the equations of statics.

Here we shall be discussing a few such cases of statically indeterminate beams :

(*i*) **Continuous beam.** In this type of beam, the beam is supported on more than two supports. The degree of indeterminacy depends on the number of supports. For the continuous beam shown in Fig. 3.11, there are three unknowns R_1, R_2 and R_3. The two equations of statics must be supplemented by one equation based on deformations.

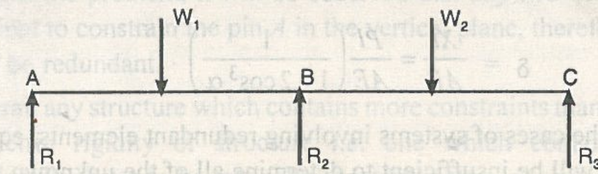


Fig. 3.11

However, when the number of supports are more, it is convenient to consider the bending moments at the various supports as unknown (rather than the reactions themselves) and write deformation equations in terms of bending moments.

(*ii*) **Fixed beam.** In this type of beam, the ends are constrained to remain in horizontal position. Due to fixidity, the slope of the beam is zero at each end, and a couple or moment is induced at each end to satisfy this condition. The induced moment M_1 and M_2 will be in the opposite direction to that of the moment due to external loading

In this case, there are four unknown : R_1, R_2, M_1 and M_2. Thus the two statics equations must be supplemented by two additional equations arising from deformations.

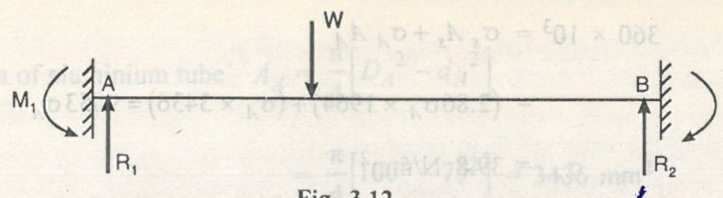


Fig. 3.12

(*iii*) **Propped cantilever.** In this case, there are three unknowns, R_1, R_2 and M. The two equations of statics *i.e.* $\Sigma V = 0$ and $\Sigma M = 0$ must be supplemented by a third equation expressing the conditions of compatibility of deformations. Since the end B does not yield, we have y_B due to $W = y'_B$ due to R_1.

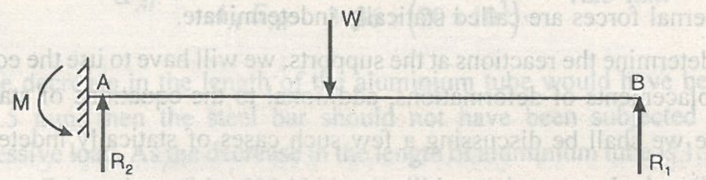


Fig. 3.13

From which

$$X = \frac{P}{1+2\cos^3\alpha} \quad \text{and} \quad Y = \frac{P\cos^2\alpha}{1+2\cos^3\alpha}$$

The total vertical deflection of point A

$$\delta = \frac{Xl}{AE} = \frac{Pl}{AE}\left(\frac{1}{1+2\cos^3\alpha}\right)$$

In all the cases of systems involving redundant elements, equations of statics will be insufficient to determine all of the unknown forces and must be supplemented by equations of consistent deformation based on the geometry of the system.

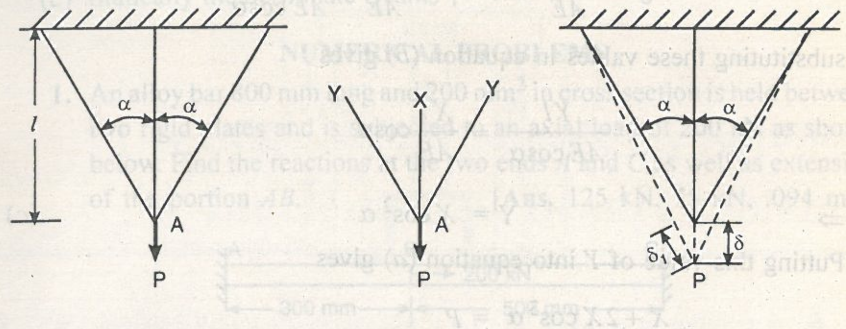
3.8 A SHORT NOTE ON CONDITION FOR STATISTICAL INDETERMINANCY

Cantilever, simply supported and overhanging beams are known as statically determinate beams because support reactions of these beams are determined by use of equation of static equilibrium.

On the other hand, propped cantilever, fixed beam and continuous beams are known as statically indeterminate beams because support reactions of these beams can not be determined by use of equation of static equilibrium.

Hence, when the equations of statics, *i.e.* $\Sigma V = 0$, $\Sigma M = 0$ and $\Sigma H = 0$ must be supplemented by another equation expressing the condition of compatibility, that state is referred as *condition for statical indeterminancy.*

Consider the simple structure made up of three tension members, each of cross sectional area A and modulus of elasticity E and subjected to a vertical load P at A. Y is the tensile force in each inclined bar and X is the tensile force in vertical bar.



(*a*) (*b*) (*c*)

Fig. 3.14

By the equation of statics :

$$X + 2Y\cos\alpha = P \qquad ...(a)$$

But this equation alone is insufficient to determine the values of X and Y uniquely. For this reason the structure is said to be statically indeterminate.

Looked at the problem, it will be observed that any two of the three bars are sufficient to constrain the pin A in the vertical plane, therefore a third bar is said to be redundant.

In general, any structure which contains more constraints than are necessary for geometric rigidity of structure *i.e.* one which contains redundant constraints will prove to be statically indeterminate.

In statics, we assume that the bars of the structure are absolutely rigid and they are undeformable. Hence by statics there would be no way to ascertain how the load P was divided among the three bars.

However, the bars are actually elastic and stretch slightly under tension. Since these are connected together by the pin A, it is evident that the amounts they stretch must be related in some way by the geometry of structure. (see Fig. 3.14*c*).

Let δ be the elongation of the vertical bar and

δ_1 be the elongation of each inclined bar.

These elongation are extremely small compared with the dimensions of the structure,

$$\delta_1 = \delta\cos\alpha \qquad ...(b)$$

This equation is known as compatibility equation of deformation. This is the key to the problem.

Using Hooke's Law

$$\delta = \frac{Xl}{AE} \text{ and } \delta_1 = \frac{Yl_1}{AE} = \frac{Yl}{AE\cos\alpha}$$

substituting these values in equation (*b*) gives

$$\frac{Yl}{AE\cos\alpha} = \frac{Xl}{AE}\cos\alpha$$

$$\Rightarrow \qquad Y = X\cos^2\alpha \qquad ...(c)$$

Putting this value of Y into equation (*a*) gives

$$X + 2X\cos^3\alpha = P$$

SUMMARY

1. When the equations of statics, *i.e.* $\Sigma V = 0$, $\Sigma H = 0$ and $\Sigma M = 0$ alone are insufficient to determine all the unknown forces and must be supplemented by another equation expressing the condition of compatibility, that state is referred as condition for statical indeterminancy.
2. Cantilever, simply supported and overhanging beams are statically determinate beams.
3. Propped cantilever, fixed beam and continuous beams are known as statically indeterminate beams because support reactions of these beams cannot be determined by use of equations of static equilibrium.
4. Any structure which contains more constraints than are necessary for geometric rigidity of structure *i.e.* one which contains redundant constraints, will prove to be statically indeterminate.
5. The degree of indeterminancy in a continuous beam depends on the number of supports.
6. For solving statically indeterminate structural problems, the deformation characteristics of the structure are also taken into account alongwith the statical equilibrium equations.
7. Equations which contain the deformation characteristics are called compatibility equations.
8. To solve a statically indeterminate problem, first we write equations of statics. After that we will consider the nature of deformation (compatibility equations) to get additional equation and finally we solve the set of equations of statics and compatibility.

REVIEW QUESTIONS

Write short notes on the following :

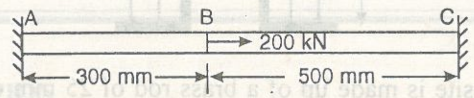
(*a*) Statically Indeterminate Structure

(*b*) Statistical Indeterminancy

(*c*) Statically Indeterminate Beams

NUMERICAL PROBLEMS

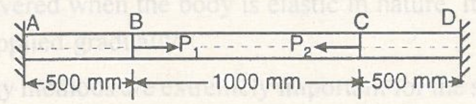
1. An alloy bar 800 mm long and 200 mm^2 in cross section is held between two rigid plates and is subjected to an axial load of 200 kN as shown below. Find the reactions at the two ends A and C as well as extension of the portion AB. **[Ans.** 125 kN, 75 kN, .094 mm]



2. A bar ABC fixed at both ends A and C is loaded by an axial load P at C. If the distances AB and BC are equal to a and b respectively then find the reactions at the ends A and C.

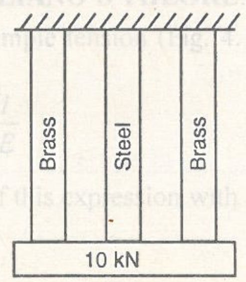
$$\left[\textbf{Ans. } \frac{Pb}{a+b} \text{ and } \frac{Pa}{a+b}\right]$$

3. A prismatic bar $ABCD$ has built-in ends A and D. It is subjected to two point loads P_1 and P_2 equal to 80 kN and 40 kN at B and C as shown below. Find the reactions at A and D. **[Ans.** 70 kN, 50kN]



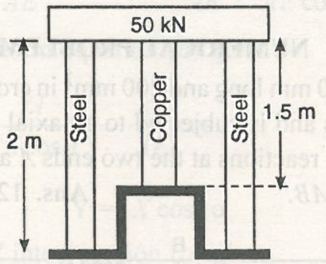
4. Three long parallel wires equal in length are supporting a rigid bar connected at their bottoms as shown in Figure below. If the cross sectional area of each wire is 100 mm^2, calculate the stresses in each wire. E_B = 100 GPa, E_S = 200 GPa.

[Ans. σ_B = 2 MPa, σ_S = 50 MPa]



5. Two steel rods and one copper rod each of 20 mm diameter, together support a load of 50 kN. Find the stresses in each rod. Take E for steel and copper as 200 GPa and 100 GPa respectively.

[**Ans.** σ_C = 39.8 MPa, σ_S = 59.7 MPa]



6. A composite is made up of a brass rod of 25 mm diameter, enclosed in a steel tube of 40 mm external diameter and 35 mm internal diameter. The ends of the rod and tube are securely fixed. Find the stresses developed in the brass rod and steel tube, when the composite bar is subjected to an axial pull of 45 kN. Take *E* for brass as 80 GPa and *E* for steel as 200 GPa. [**Ans.** 36.6 MPa, 91.5 MPa]

4

Use of Energy Methods for Solving Indeterminate Beam Problems

4.1 INTRODUCTION

In earlier chapters we have studied about different types of problems of statically indeterminate structures. Till now, the problems of stresses and strains at a point were related through the constitutive equations. During those problems, the shape or size of the body as a whole was not considered.

Here, we shall consider the entire body or structural element, alongwith the forces on it. Hooke's law will relate the force acting on the body to the displacement when the body deforms under the action of the externally applied forces, the work done by these forces is stored as strain energy inside the body, which can be recovered when the body is elastic in nature. It is assumed that these forces are applied gradually.

The strain energy methods are extremely important for the solution of many problems in structural analysis.

The theorem of Castigliano is very useful in the treatment of statically indeterminate problems. In earlier pages we have studied about the Castigliano's theorem, however for the continuity of the subject matter let us go through a brief review of this theorem.

4.2 REVIEW OF CASTIGLIANO'S THEOREM

For a prismatic bar under simple tension (Fig. 4.1) the strain energy is

$$U = \frac{P^2 l}{2AE}$$

By taking the derivative of this expression with respect to the applied load *P*, we obtain

$$\frac{dU}{dP} = \frac{Pl}{AE} = \delta$$

Thus the derivative of the strain energy with respect to the applied load P gives the deflection of its point of application in the direction of the load.

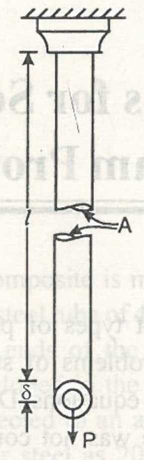


Fig. 4.1

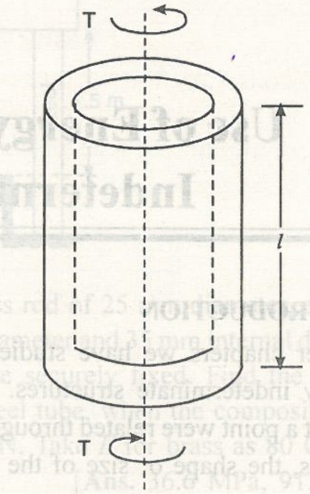


Fig. 4.2

Now consider a shaft of circular cross-section subjected to torsion as shown in Fig. 4.2, the strain energy is

$$U = \frac{T^2 l}{2GJ}$$

The derivative of this expression with respect to the applied torque T becomes

$$\frac{dU}{dT} = \frac{Tl}{GJ} = \phi$$

which is the angle of twist of one end of the shaft with respect to the other.

Let us now consider a cantilever beam bent by a transverse load P at the free end (Fig. 4.3), the strain energy of bending is

$$U = \frac{P^2 l^3}{6EI}$$

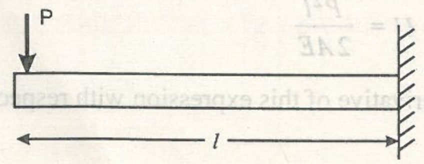


Fig. 4.3

The derivative of this expression with respect to P is

$$\frac{dU}{dP} = \frac{Pl^3}{3EI} = \delta$$

which is seen to be the deflection of the end of the beam in the direction of the applied load P.

Each of these cases is simply an example of Castigliano's theorem. The partial derivative of the strain energy, represented as a quadratic function of the forces, with respect to any one of these forces, gives the corresponding component of displacement of the point of application of that force.

Now, before discussing the case of statically indeterminate structure let us solve the case of a statically determinate beam problem.

Example 4.1 A simple supported beam with overhang is loaded as shown in Fig. 4.4. Using theorem of Castigliano's, find the vertical deflection of point C.

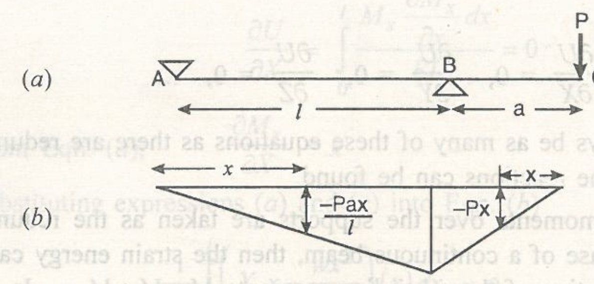


Fig. 4.4

Solution. The bending moment diagram for the beam is shown in Fig. 4.4(b). Between A and B, a general expression for bending moment at any distance x to the right of A is

$$M_x = -\frac{Pax}{l}$$

Between B and C, the bending moment at any distance x to the left of C is

$$M_x = -Px$$

Now $$U = \int_0^l \frac{M_x^2 dx}{2EI}$$

Hence, the strain energy of bending in the beam becomes

$$U = \int_0^l \frac{P^2 a^2 x^2}{2EIl^2} dx + \int_0^a \frac{P^2 x^2}{2EI} dx = \frac{P^2 a^2}{6EI} \ldots$$

Now $$\delta_n = \frac{\partial U}{\partial P_n}$$

Hence $$\delta = \frac{\partial U}{\partial P} = \frac{Pa^2}{3EI}(l+a)$$

4.3 APPLICATION OF CASTIGLIANO'S THEOREM TO STATICALLY INDETERMINATE PROBLEMS

Consider the case of a continuous beam having several redundant supports. Denoting by $X, Y, Z, \ldots$ the statically indeterminate reactions, the strain energy in the beam will be a function of these forces. The displacements of their points of application will be obtained as $\frac{dU}{dX}, \frac{dU}{dY}, \frac{dU}{dZ}, \ldots$

In accordance to the conditions of constraint, all these displacements are known to be zero.

Hence we have

$$\frac{\partial U}{\partial X} = 0, \quad \frac{\partial U}{\partial Y} = 0, \quad \frac{\partial U}{\partial Z} = 0, \ldots$$

There will always be as many of these equations as there are redundant reactions so that the reactions can be found.

If the bending moments over the supports are taken as the redundant quantities in the case of a continuous beam, then the strain energy can be expressed as a function of these bending moments $M_1, M_2, M_3, \ldots$ In such case, the partial derivatives $\frac{\partial U}{\partial M_1}, \frac{\partial U}{\partial M_2}, \frac{\partial U}{\partial M_3}, \ldots$ will represent the relative rotations between tangents to the elastic line on the two sides of each support. However, from the continuity of the elastic line over each support, we know that these relative rotations are all zero.

Hence again $$\frac{\partial U}{\partial M_1} = 0, \quad \frac{\partial U}{\partial M_2} = 0, \quad \frac{\partial U}{\partial M_3} = 0$$

In general, to find the redundant forces in a statically indeterminate system, we remove the redundant constraints and replace them by the corresponding forces. Then expressing the strain energy of the system in terms of the forces and applying the Castigliano's theorem, we obtain equations from which, the redundant forces can be calculated.

Illustration. Consider the case of a propped cantilever beam under uniform load (Fig. 4.5).

This system has one statically indeterminate reaction.

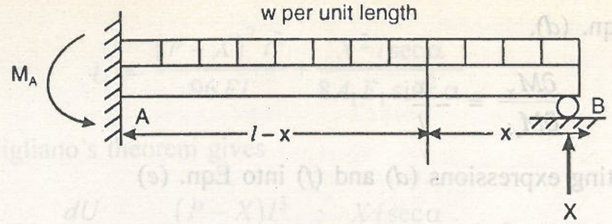


Fig. 4.5

Choosing the vertical reaction X at B as the redundant force, the bending moment at any point along the beam will be

$$M_x = X.x - \frac{wx^2}{2} \qquad \ldots(a)$$

The strain energy is

$$U = \int_0^l \frac{M_x^2 dx}{2EI}$$

$$\therefore \quad \frac{\partial U}{\partial X} = \int_0^l \frac{M_x \frac{\partial M_x}{\partial x} dx}{EI} = 0 \qquad \ldots(b)$$

From Eqn. (a), $$\frac{\partial M_x}{\partial X} = x \qquad \ldots(c)$$

Substituting expressions (a) and (c) into Eqn. (b)

$$\frac{1}{EI}\int_0^l \left(X.x - \frac{wx^2}{2}\right)(x)dx = 0$$

or $$\frac{Xl^3}{3} - \frac{wl^4}{8} = 0$$

or $$X = \frac{3}{8}wl$$

Choosing the restraining couple M_A at the built-in end of the beam as the redundant reaction, the bending moment at any cross-section becomes

$$M_x = \frac{wlx}{2} - \frac{M_A x}{l} - \frac{wx^2}{2} \qquad \ldots(d)$$

Since the angle of rotation of the tangent at A is known to be zero, the Castigliano's theorem gives

$$\frac{\partial U}{\partial M_A} = \int_0^l \frac{M_x \frac{\partial M_x}{\partial M_A} dx}{EI} = 0 \qquad \ldots(e)$$

From Eqn. (*d*),

$$\frac{\partial M_x}{\partial M_A} = -\frac{x}{l} \qquad \ldots(f)$$

Substituting expressions (*d*) and (*f*) into Eqn. (*e*)

$$\frac{1}{EI}\int_0^l \left(\frac{wlx}{2} - \frac{M_A x}{l} - \frac{wx^2}{2}\right)\left(-\frac{x}{l}\right)dx = 0$$

$$\Rightarrow \qquad M_A = \frac{wl^2}{8}$$

Example 4.2 To reduce deflection, a simply supported wooden beam *AB*, loaded at the middle, is trussed by steel cable *AD* and *BD* and a post *CD* arranged as shown in Fig. 4.6.

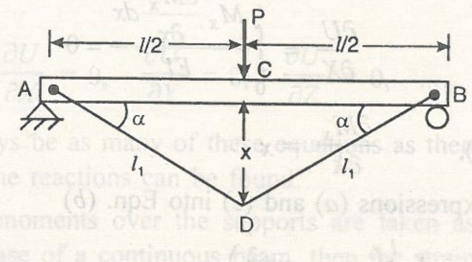


Fig. 4.6

Neglecting axial shortening of both the beams and the post, find the compressive force X induced in the post. The beam has flexural rigidity EI and the cables have cross sectional area A_1 and modulus of elasticity E_1.

Solution. The net downward load on the middle of the beam is $Q = P - X$ and the tension in each cable is

$$S_1 = \frac{X}{2\sin\alpha}$$

Neglecting strain energy of compression in the beam and post, the strain energy of the system is

$$U = \frac{Q^2 l^3}{96EI} + 2\left(\frac{S_1^2 l_1}{2A_1E_1}\right)$$

Substituting for Q and S_1 and noting that

$$l_1 = \frac{1}{2} l \sec\alpha \text{, this becomes}$$

$$U = \frac{(P-X)^2 l^3}{96EI} + \frac{X^2 l \sec\alpha}{8A_1E_1\sin^2\alpha}$$

The Castigliano's theorem gives

$$\frac{dU}{dX} = \frac{(P-X)l^3}{48EI} + \frac{X l \sec\alpha}{4A_1E_1\sin^2\alpha} = 0$$

From which

$$X = \frac{P}{1 + \dfrac{12\sec\alpha}{l^2\sin^2\alpha}\dfrac{EI}{A_1E_1}}$$

Example 4.3 Two wooden beams of identical cross-section are supported at their ends and cross at their mid-points as shown in Fig. 4.7.

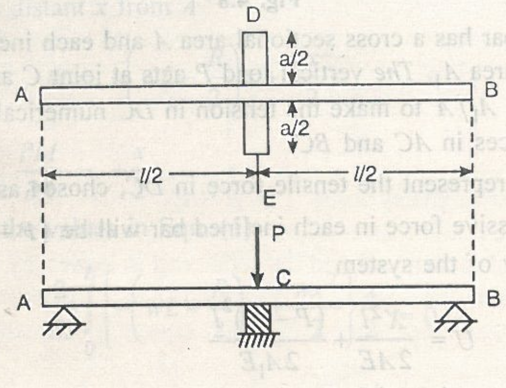


Fig. 4.7

When unloaded, they are just in contact at *C*. What interactive force X will exist between the two beams at *C* when a vertical load P is applied to the upper beam as shown ?

Solution. The net downward load on the beam *AB* is $P - X$, while that on the beam *DE* is X. The total strain energy in a simple beam loaded at the middle by a force Q is

$$U = \frac{Q^2 l^3}{96EI}$$

Thus the strain energy in the two beams becomes

$$U = \frac{(P-X)^2 l^3}{96EI} + \frac{X^2 a^3}{96EI}$$